Effect of the Interconnected Network Structure on the Epidemic Threshold

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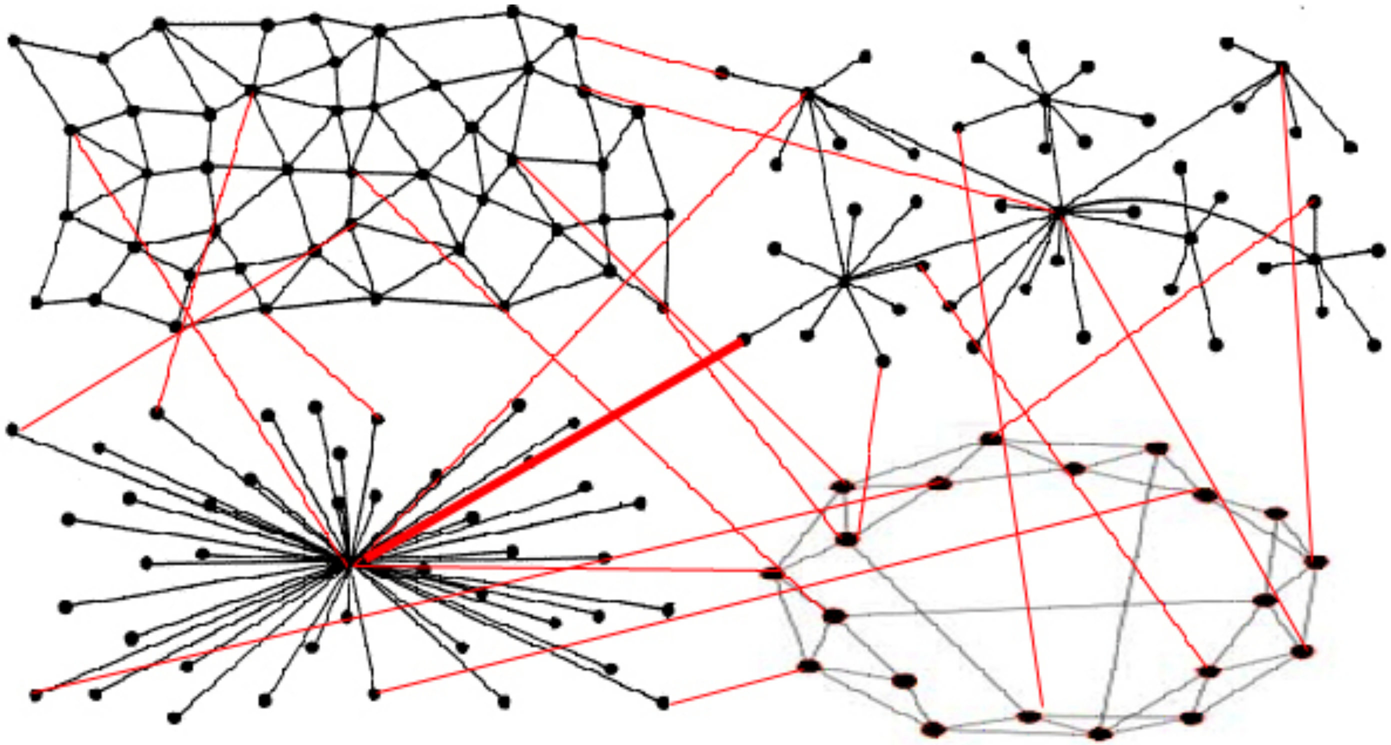
Catastrophic Cascading Failures in Interdependent Networks

Nature, 464, 1025 (2010)
PRL ,105, 0484 (2010)
PNAS, 108, 1007 (2011)
PRL, 107, 195701 (2011)
Nature Phys., 8, 40 (2012)
…



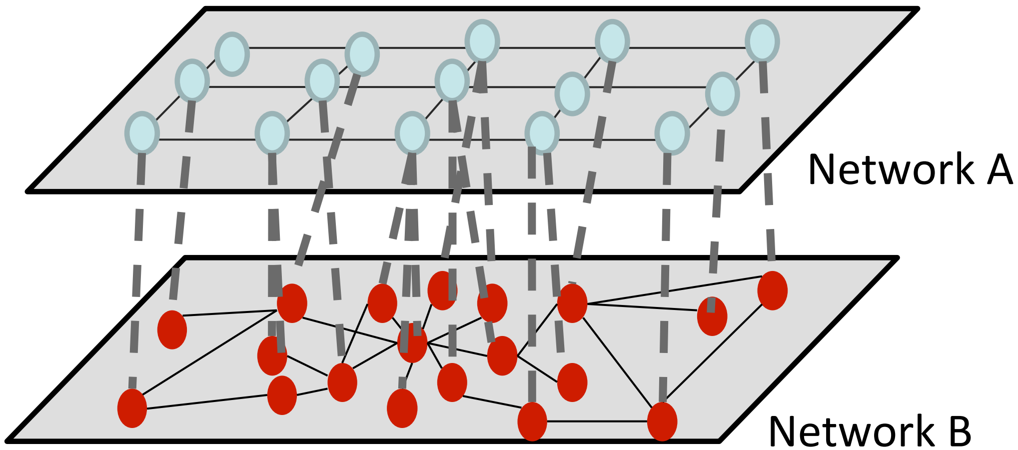
Electric grid,
Communication
Transportation …..

Two types of links:
1. Connectivity
2. Dependency

Cascading disaster

Raissa D'sousa-same type

Non-Consensus model in interconnected networks



Opinion interaction rules within each individual network may differ from those between the networks

Q. Li, L. A. Braunstein, H. Wang, J. Shao, H. E. Stanley and S, Havlin
Non-consensus Opinion Models on Complex Networks, J Stat Phys, 2012

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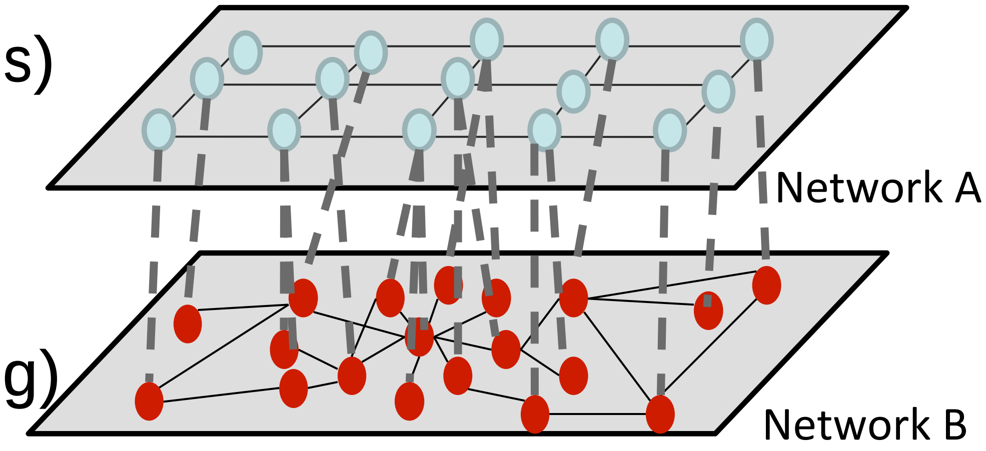
Epidemic spreading in interconnected networks

Why Epidemics?

- self-replicating objects(worms)
- propagation errors
- rumors(social nets)
- epidemic algorithms(gossiping)
- cybercrime: network robustness & security

Why Interconnected networks?



Epidemics may spread across Multiple networks (species, communities)

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Epidemic spreading in interconnected networks

Recent studies:

Phys. Rev. E 85 , 066109 (2012): SIR epidemics may infect none, one or both networks.

Phys. Rev. E 86 , 026106 (2012): in SIS model, an endemic state may appear though an epidemic is unable to propagate in each single network.

arxiv.org/pdf/1212.4194 (2012): SIS epidemic threshold on generic interconnected networks.

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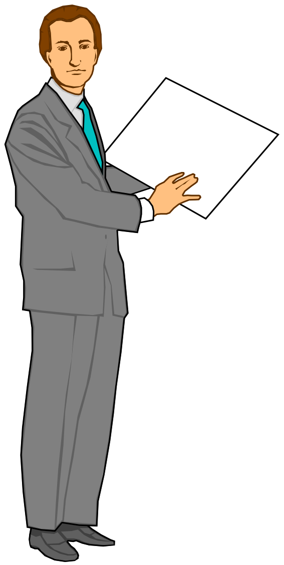
Outline



SIS epidemic model in a single network

SIS model in interconnected networks

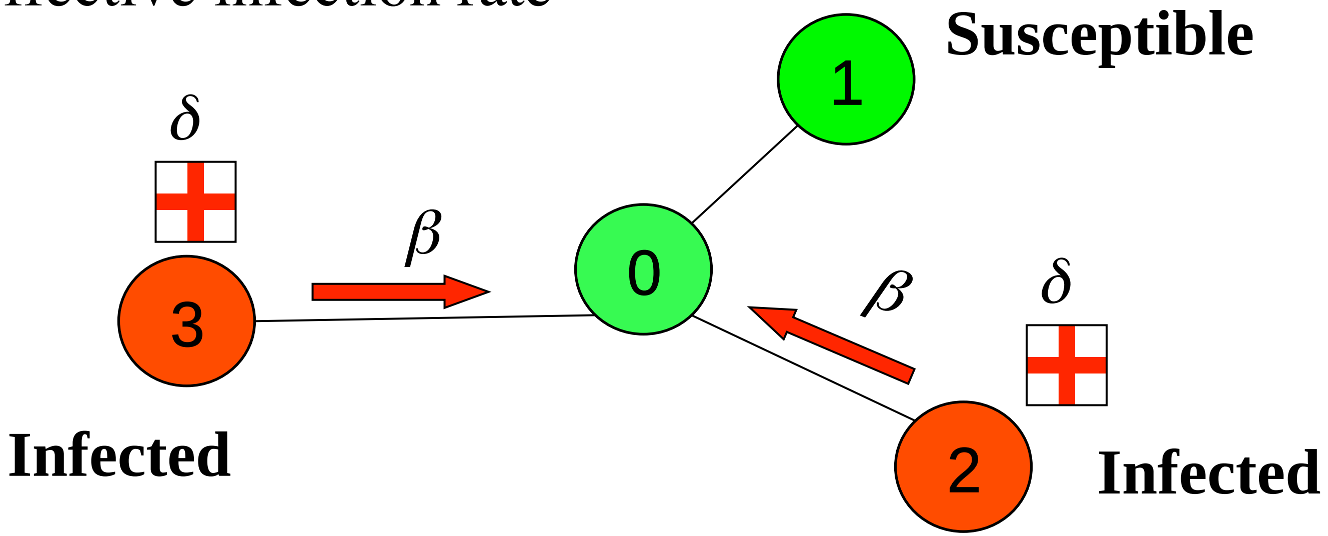
Effect of interconnected network structure on the epidemic threshold

Conclusion

Susceptible-Infected-Susceptible (SIS) model in a single network

- Homogeneous birth (infection) rate β on all edges between infected and susceptible nodes
- Homogeneous death (curing) rate δ for infected nodes

$\tau = \beta / \delta$: effective infection rate



Infection and curing are independent Poisson processes

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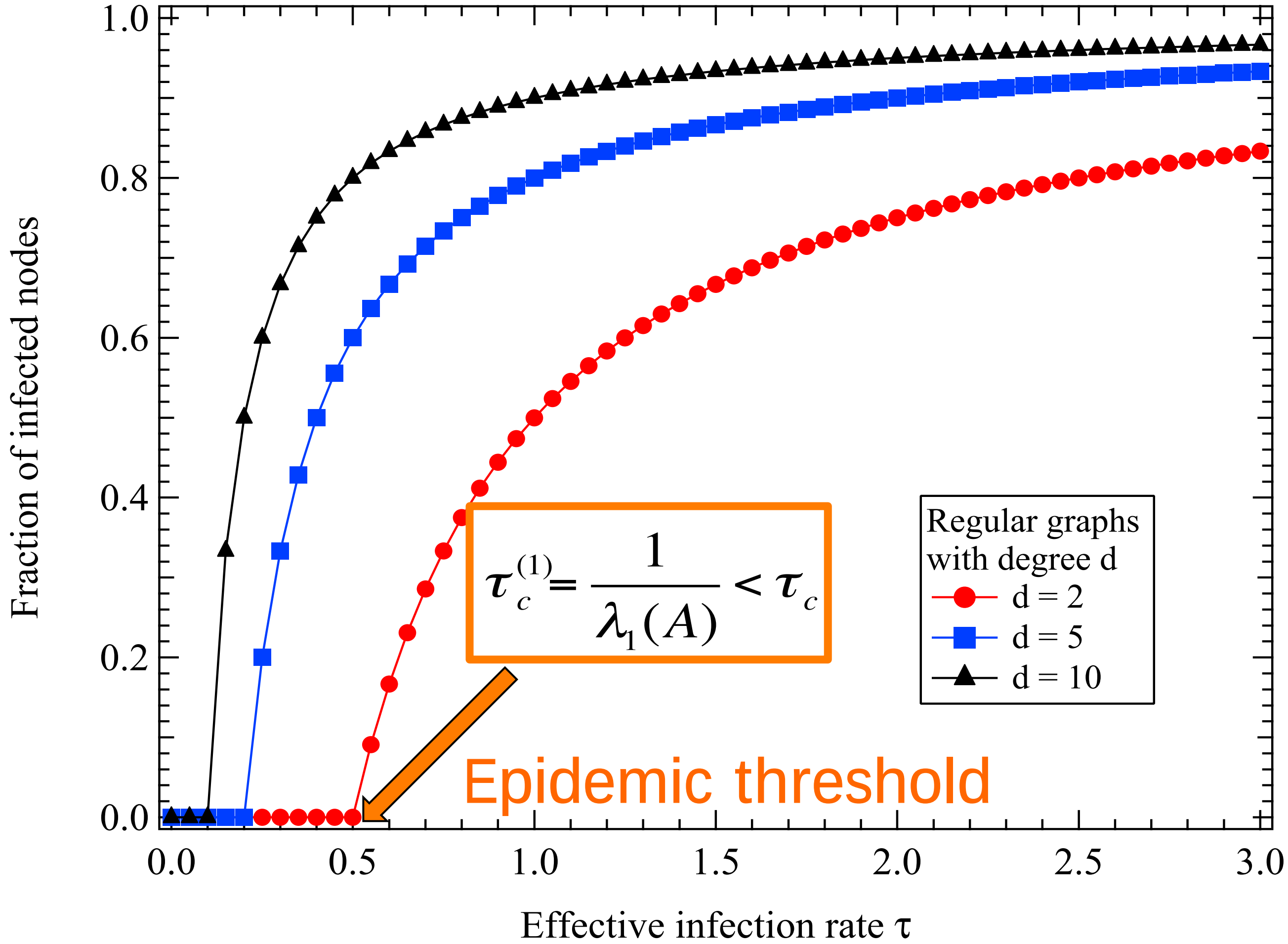
SIS epidemic threshold in a single network

β : infection rate per link

δ : curing rate per node

τ= β/ δ : effective infection rate



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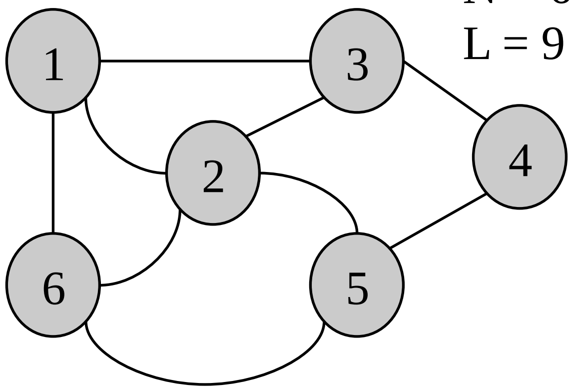
Epidemic threshold $1/\lambda_1(A)$



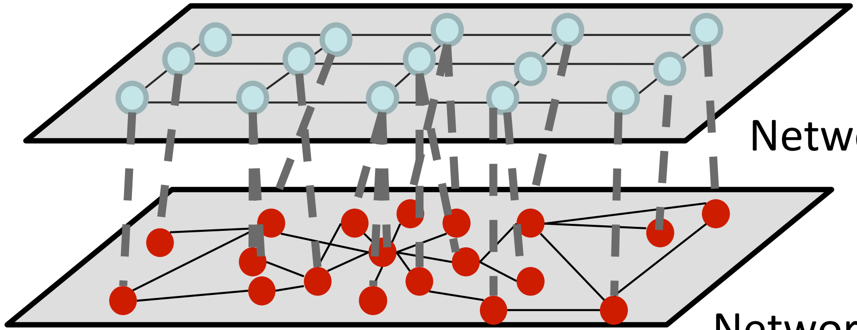
N = 6
L = 9

$$A_{N\times N} = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 \end{bmatrix} = A^T$$

$$A = X \Lambda X^T$$

$$\lambda_N \le \lambda_{N-1} \le \cdots \le \lambda_2 \le \lambda_1$$

SIS model in interconnected networks



$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}, B = \begin{bmatrix} 0 & B_{12} \\ B_{12}^T & 0 \end{bmatrix}$$

Two types of links:
Connectivity links within each single network
Interconnections

β : infection rate at connectivity link

δ : curing rate per node

α β: infection rate at interconnection link

$\tau = \beta / \delta$

$$\tau_c = \frac{1}{\lambda_1 (A + \alpha B)}$$

Effect of interconnected network structure on $\lambda_1(A+\alpha B)$

$$\lambda_1\left(A+\alpha B\right)=f\left(\alpha,\lambda_1\left(A_1\right),\lambda_1\left(A_2\right),\lambda_1\left(B_{12}\right),A_1,A_2,B_{12},x,y\right)?$$

x and y: the principal eigenvector of A_1 and A_2 respectively

Three approaches:

1. When AB=BA,

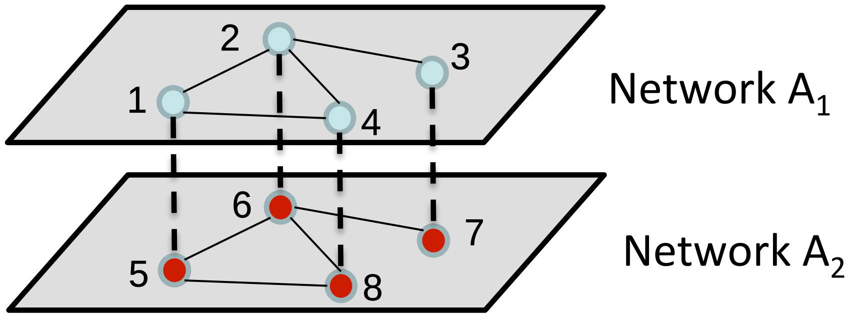
$$\lambda_1\left(A+\alpha B\right)=\lambda_1\left(A\right)+\alpha\lambda_1\left(B\right)=\max(\lambda_1\left(A_1\right),\lambda_1\left(A_2\right))+\alpha\lambda_1\left(B\right)$$

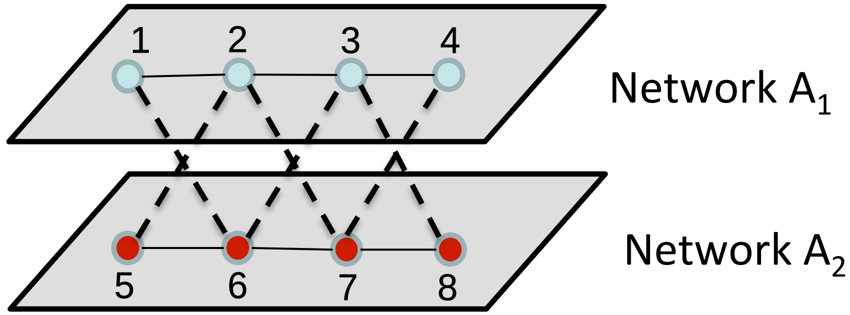
2. Perturbation approximation for small and large α
3. Lower and upper bound of $\lambda_1(A+\alpha B)$

Effect of interconnected network structure on $\lambda_1(A+\alpha B)$

Approach 1: When AB=BA,

$$\lambda_1(A+\alpha B)=\lambda_1(A)+\alpha\lambda_1(B)=\max(\lambda_1(A_1),\lambda_1(A_2))+\alpha\lambda_1(B)$$





$$A=\begin{bmatrix} A_1 & 0 \\ 0 & A_1 \end{bmatrix}, B=\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$$

$$\lambda_1(A+\alpha B)=\lambda_1(A_1)+\alpha$$

$$A=\begin{bmatrix} A_1 & 0 \\ 0 & A_1 \end{bmatrix}, B=\begin{bmatrix} 0 & A_1 \\ A_1 & 0 \end{bmatrix}$$

$$\lambda_1(A+\alpha B)=(1+\alpha)\lambda_1(A_1)$$

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Effect of interconnected network structure on $\lambda_1(A+\alpha B)$

Approach 2: Perturbation approximation for small and large α for both the case $\lambda_1(A_1)=\lambda_1(A_2)$ and $\lambda_1(A_1)\neq\lambda_1(A_2)$.

Theorem: For small α, when $\lambda_1(A_1)=\lambda_1(A_2)$,

$$\lambda_1(A+\alpha B)=\lambda_1(A_1)+\alpha x^T B_{12} y/2+O(\alpha^2)$$

x and y: the principal eigenvector of A_1 and A_2 respectively

$$x^T B_{12} y=\sum_{i,j} B_{12}(i,j)x_i y_j$$

Effect of interconnected network structure on $\lambda_1(A+\alpha B)$

Approach 3: Lower and upper bounds

Theorem 8 *The best possible lower bound* $\lambda_1^2(W) \geq \frac{z^T W^2 z}{z^T z}$ *by choosing* z *as the linear combination of* x *and* y*, the largest eigenvector of* A_1 *and* A_2 *respectively, is*

$$\lambda_1^2(W) \geq \frac{\left(\lambda_1^2(A_1) + \alpha^2 \left\|B_{12}^T x\right\|_2^2 + \lambda_1^2(A_2) + \alpha^2 \left\|B_{12} y\right\|_2^2\right)}{2} + \sqrt{\left(\frac{\lambda_1^2(A_1) + \alpha^2 \left\|B_{12}^T x\right\|_2^2 - \lambda_1^2(A_2) - \alpha^2 \left\|B_{12} y\right\|_2^2}{2}\right)^2 + \theta^2} \quad (3)$$

where $\theta = \alpha\left(\lambda_1(A_1) + \lambda_1(A_2)\right) x^T B_{12} y$

$$x^T B_{12} y = \sum_{i,j} B_{12}(i,j) x_i y_j$$

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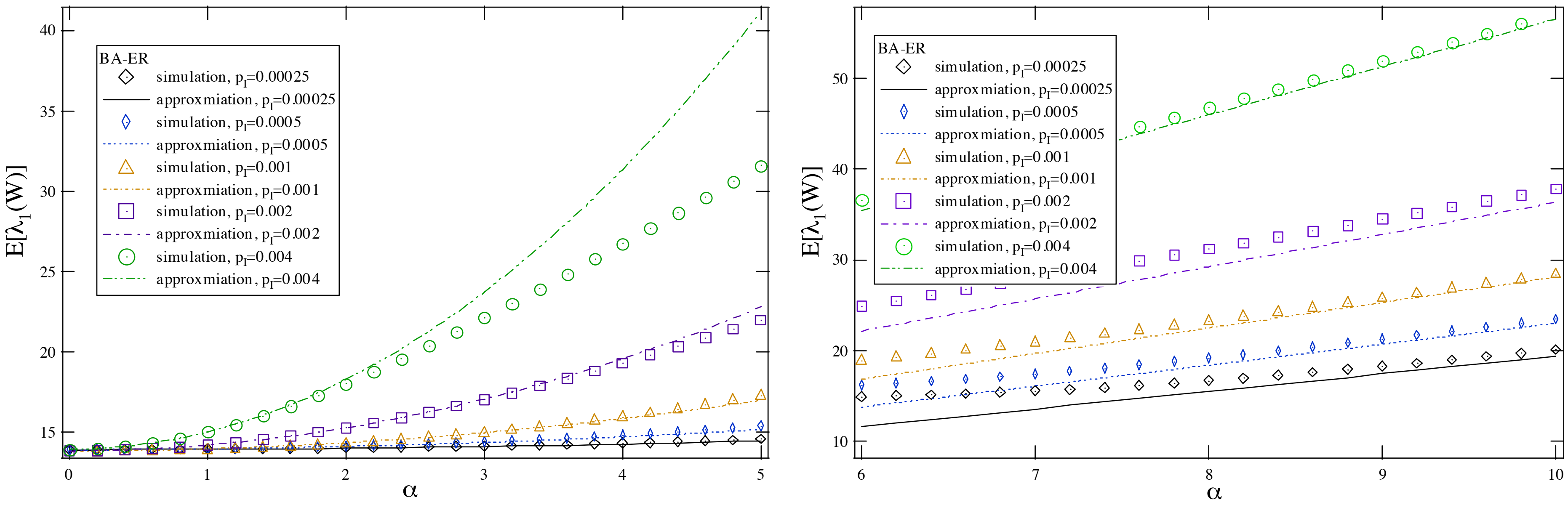
Simulation verifications



(a) (b)

Both single network BA and ER have N=1000 nodes and <k>=6

Interconnections are placed randomly with density p_I

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Conclusion

- The epidemic threshold of two interconnected networks is $1/\lambda_1(A+\alpha B)$
- We analytically derive $\lambda_1(A+\alpha B)$ using perturbation approximation for small and large α, its lower and upper bound as such function

$$f\left(\alpha,\lambda_1\left(A_1\right),\lambda_1\left(A_2\right),\lambda_1\left(B_{12}\right),A_1,A_2,B_{12},x,y\right)$$

- These approximations and bounds are valid for any interconnected network structure and reveal how component network properties affect the epidemic threshold.

Conclusion

- The term $x^T B_{12} y = \sum_{i,j} B_{12}(i,j) x_i y_j$ contributes positively to $\lambda_1(A+\alpha B)$ suggesting a lower epidemic threshold if two nodes i and j with a larger eigenvector component product $x_i y_j$ are interconnected.

	11random	11assortative	11disassortative	11vec	11vec_g	vec	vec_g
BA-BA	12.53	12.89	12.46	12.95	12.99	24.75	27.15
BA-ER	12.04	12.08	12.04	12.09	12.09	23.64	24.64
ER-ER	7.98	8.10	7.86	8.12	8.13	21.05	23.52

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Thank You

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