Phase Transitions on Networks :
- synchronization as an example

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2014 FIFA World Cup Brazil™ ALL IN ONE RHYTHM™ 12 June

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QUALIFIERS

Asia

News | Matches | Groups and Standings | Teams | Format | Statistics | Photos

Round 1 | Round 2 | Round 3 | Round 4 | Round 5 | Intercontinental Play-Offs

GROUP A

Team	MP	W	D	L	GF	GA	Pts
Uzbekistan	6	3	2	1	6	4	11
Korea Republic	5	3	1	1	11	5	10
Iran	5	2	1	2	2	2	7
Qatar	6	2	1	3	4	7	7
Lebanon	6	1	1	4	2	7	4

Details | News | Analysis | Matches

11/09/12 18:00	Tashkent	Uzbekistan	2:2 (1:1)	Korea Republic
11/06/13 20:00	Seoul	Korea Republic	-	Uzbekistan

Team	MP	W	D	L	GF	GA	Pts
Japan	6	4	1	1	14	4	13
Jordan	6	2	1	3	6	12	7
Australia	5	1	3	1	6	6	6
Oman	6	1	3	2	6	9	6
Iraq	5	1	2	2	4	5	5

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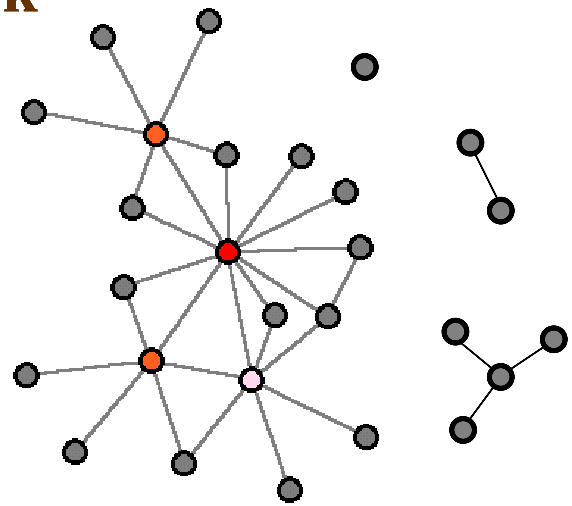
Outline

- Phase transition models on Networks
  - Ising model, Epidemic spreading, Synchronization, .. [MF]
- Phase transitions on Complete Graph (all-to-all connections) [`MF`]
- Phase transitions on Sparse networks
  - ER Random networks & Scale-free networks ["MF"]
- **Annealed approximation** for **quenched** networks (annealed networks) [^MF^]
  - Issue: validity of **annealed approximation** with quenched link disorder
    * transition point shift due to local correlations generated by quenched disorder.
    * transition nature ?? [MF regime: quenched disorder usually irrelevant]
  - (example) Kuramoto model for synchronization
    * intrinsic frequency distribution: unimodal/uniform/bimodal/…
    * transition nature changes ["MF" $\neq$ ^MF^] even for ER networks.
  - Ising model, Contact process, …
    * finite-size scaling changes for highly heterogeneous SF networks!
- Remarks

Phase transition models on Networks

Network



- nodes and links
- **no** distance information (∞ dim.)
- N : number of nodes
- k_i : degree of the i-th node
- $P(k)$: degree distribution

Adjacency matrix

$$a_{ij} = \begin{cases} 1 & \text{when nodes } \ i,j \text{ are linked,} \\ 0 & \text{when nodes } \ i,j \text{ are not linked.} \end{cases}$$

- Ising model $\quad H[\{s\}] = -J\sum_{i,j} a_{ij} s_i s_j \text{ with } s_i = \pm 1$
- Synchronization $\quad \dot{\phi}_i = \omega_i - K\sum_j a_{ij} \sin(\phi_i - \phi_j)$

Complete graph

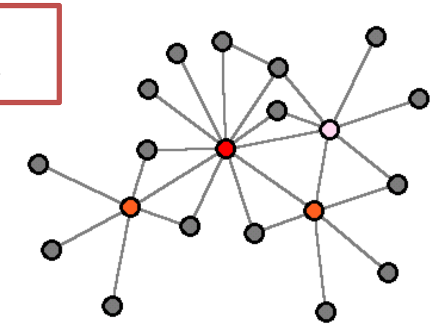
$a_{ij} = 1 \text{ for all } ij.$ $\sum_{i,j} a_{ij} \sim N^2$

Rescaling the coupling constant by N

Ising model $H[\{s\}] = -\frac{J}{N}\sum_{i,j} s_i s_j$

Kuramoto model $\dot{\phi}_i = \omega_i - \frac{K}{N}\sum_j \sin(\phi_i - \phi_j)$

Easy to solve analytically and Mean-field (`MF`) result.



Sparse networks

$\sum_{i,j} a_{ij} \sim N$

ER networks $P(k) \sim e^{-k}$

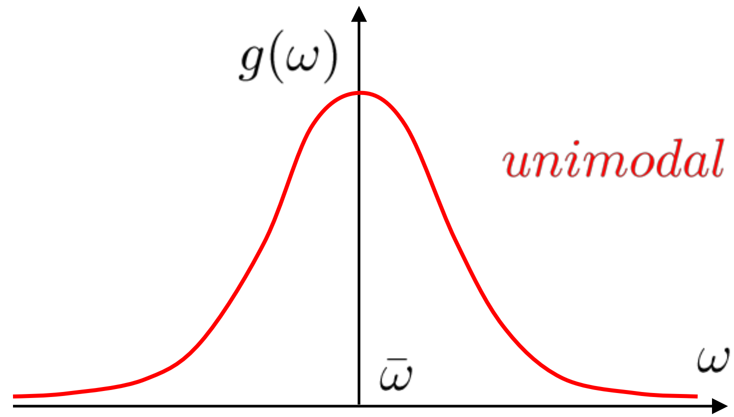
Scale-free networks $P(k) \sim k^{-\gamma}$

Quenched disorder in link configuration a_{ij}

Difficult to solve analytically - replica tricks/numerics -- "MF"
ER networks : "MF" ~ `MF` ??
Annealed approximations -- ^MF^ , and ^MF^ ~ "MF" ??

Example: Synchronization

$$\frac{d\phi_i}{dt} = \omega_i - K \sum_{j=1}^{N} a_{ij} \sin\left(\phi_i - \phi_j\right)$$



Random frequency : unimodal dist.

$$\langle \omega_i \rangle = \bar{\omega} \quad \text{and} \quad \langle \omega_i \omega_j \rangle = 2\sigma\delta_{ij}$$

Kuramoto model on CG

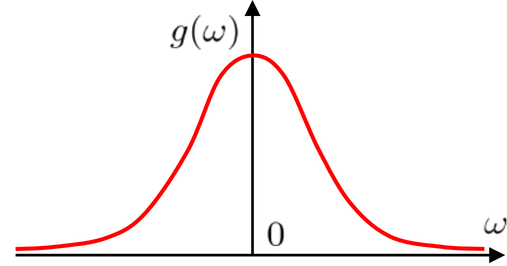
$$\frac{d\phi_i}{dt} = \omega_i - \frac{K}{N} \sum_{j=1}^{N} \sin\left(\phi_i - \phi_j\right)$$

low K

Synchronization transition

high K

Kuramoto model on CG



$$\frac{d\phi_i}{dt} = \omega_i - \frac{K}{N}\sum_{j=1}^{N} \sin\left(\phi_i - \phi_j\right)$$

$$\langle \omega_i \rangle = 0 \quad \text{and} \quad \langle \omega_i \omega_j \rangle = \delta_{ij}$$

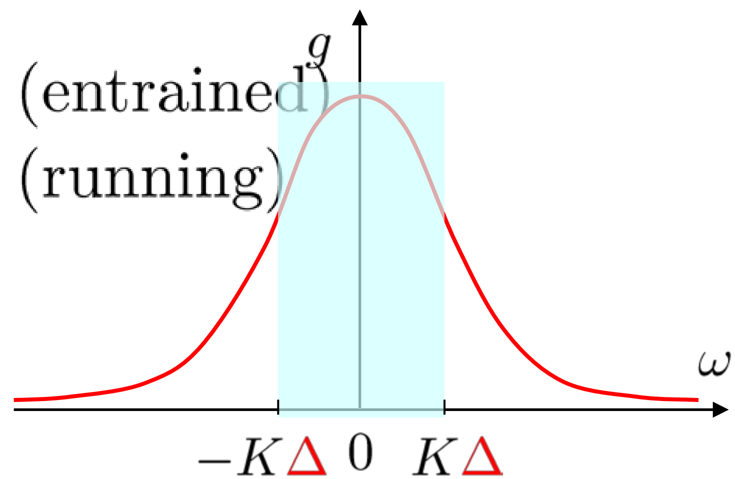
Phase Order parameter

$$\Delta e^{i\theta} \equiv \frac{1}{N}\sum_{j=1}^{N} e^{i\phi_j}$$

$$\frac{d\phi_i}{dt} = \omega_i - K\Delta \sin(\phi_i - \theta)$$

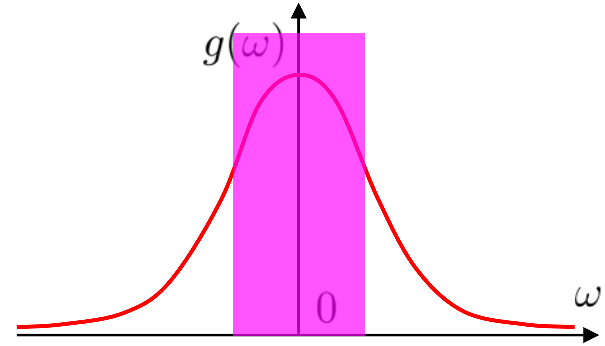
Steady state (long-time limit)

$$\overline{\frac{d\phi_i}{dt}} = \overline{\omega_i} = \begin{cases} 0, & |\omega_i| < K\Delta \quad \text{(entrained)} \\ \sqrt{\omega_i^2 - (K\Delta)^2}, & |\omega_i| > K\Delta \quad \text{(running)} \end{cases}$$



$$\Delta = \int_{|\omega| < K\Delta} d\omega\, g(\omega) \sqrt{1 - (\omega / K\Delta)^2}$$

Self-consistency equation

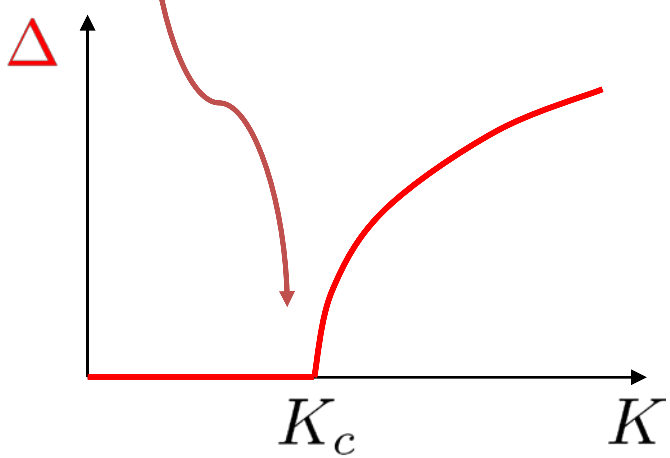


$$\Delta = \int_{|\omega|<K\Delta} d\omega g(\omega)\sqrt{1-(\omega/K\Delta)^2}$$
$$\approx aK\Delta + c(K\Delta)^3 + \mathcal{O}\left((K\Delta)^5\right) \quad [a \sim g(0), c \sim g''(0) < 0]$$

$$\Delta \sim (K-K_c)^{\beta} \quad (\beta = 1/2)$$

$$K_c = 1/a = 2/(\pi g(0))$$

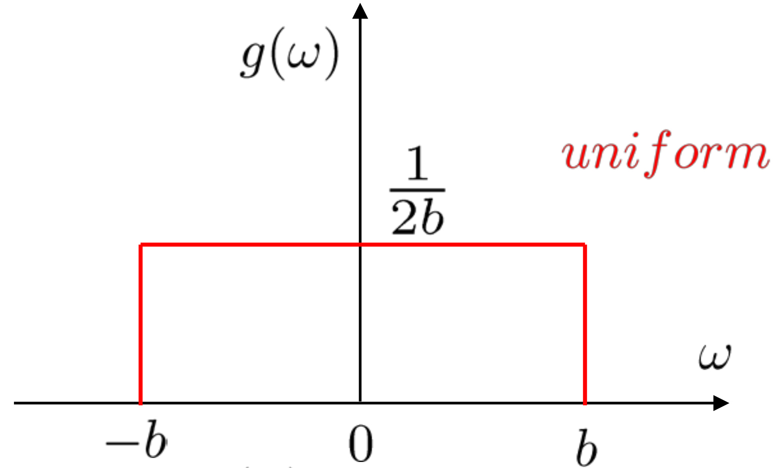
Mean field (MF) scaling



- slow & gradual entrainment of oscillators with small frequencies to large frequencies .

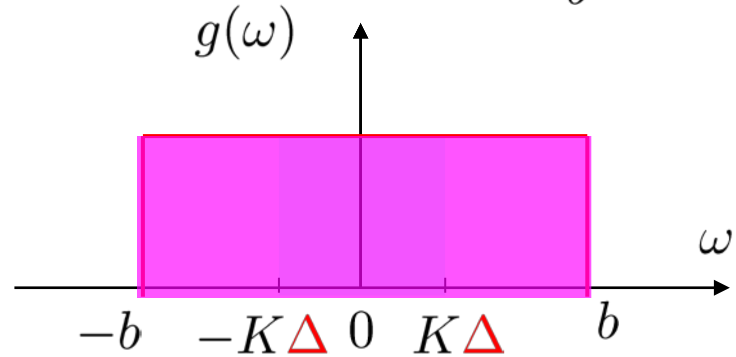
Kuramoto model on CG

$$\frac{d\phi_i}{dt} = \omega_i - \frac{K}{N}\sum_{j=1}^{N} \sin(\phi_i - \phi_j)$$



Random frequency : uniform dist.

$$\Delta = \int_{|\omega|<K\Delta} d\omega g(\omega)\sqrt{1-(\omega/K\Delta)^2}$$



(i) $|K\Delta| \leq b$

$$\Delta = \frac{1}{2b}\int_{-K\Delta}^{K\Delta} d\omega\sqrt{1-(\omega/K\Delta)^2} = \frac{\pi K\Delta}{4b} \Longrightarrow \Delta = 0$$

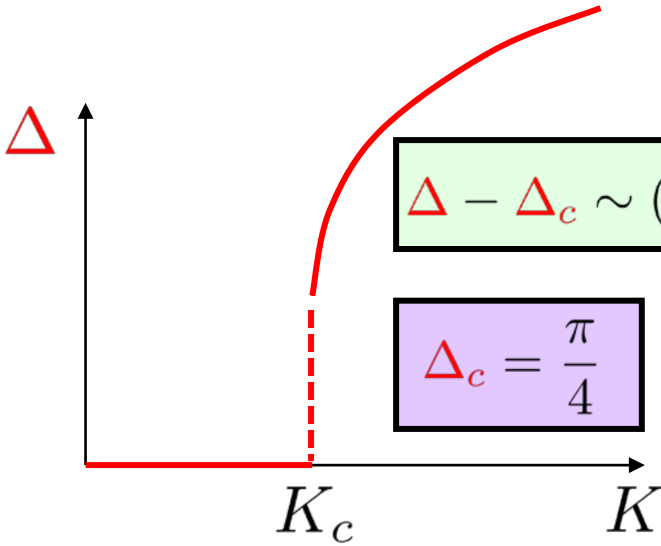
- sudden & full entrainment of oscillators
- Why? High-freq. oscillators are strong enough to destabilize synchronization of low-freq. oscillators.

(ii) $K\Delta \geq b$

$$\Delta = \frac{1}{2b}\int_{-b}^{b} d\omega \sqrt{1-(\omega/K\Delta)^2} = \frac{K\Delta}{2b} f(b/K\Delta)$$

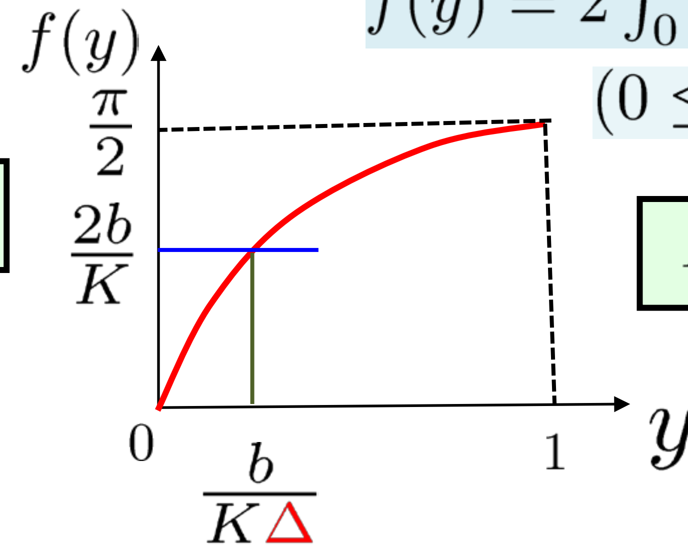
$$f(y) = 2\int_0^y dx\sqrt{1-x^2}$$

$$(0 \leq y \leq 1)$$



$\boxed{\Delta - \Delta_c \sim (K-K_c)^{2/3}}$

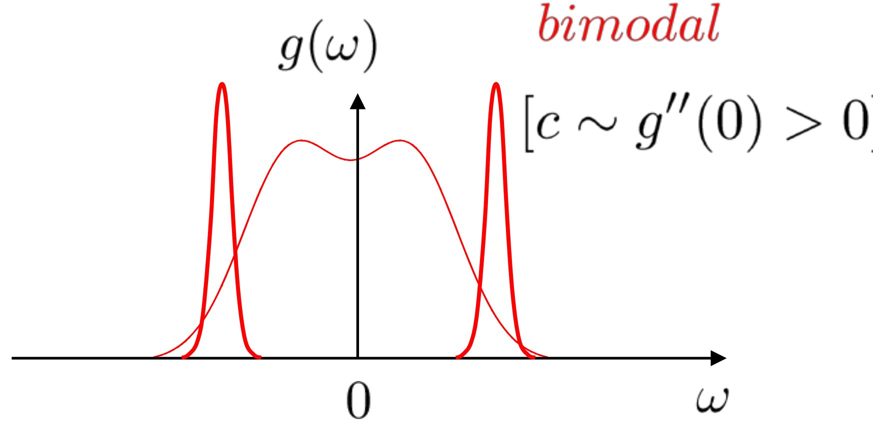
$\boxed{\Delta_c = \frac{\pi}{4}}$



$\boxed{K_c = 4b/\pi}$

- 1st order jump

Random frequency : bimodal dist.

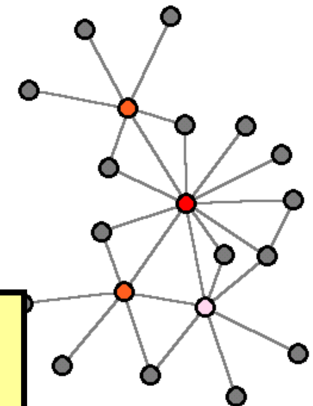


- 1st order jump + group dynamics

Kuramoto model on Sparse networks

- Many people use the uniform freq. distribution on quenched networks, just for convenience, in numerical simulations.
- Seems finding a continuous & ordinary MF transition. How?
- Different from the CG result !
- What would be the result in "annealed" approximations? Ordinary MF?
- Same results on "quenched" networks?

$$\frac{d\phi_i}{dt} = \omega_i - K \sum_{j=1}^{N} a_{ij} \sin\left(\phi_i - \phi_j\right)$$



$$\Delta e^{i\theta} \equiv \frac{1}{N} \sum_{j=1}^{N} e^{i\phi_j}$$

Local fields

$$h_i e^{i\theta_i} \equiv \sum_{j=1}^{N} a_{ij} e^{i\phi_j}$$

$$\frac{d\phi_i}{dt} = \omega_i - K h_i \sin(\phi_i - \theta_i)$$

Kuramoto model on Sparse networks

$$h_i e^{i\theta_i} \equiv \sum_{j=1}^{N} a_{ij} e^{i\phi_j}$$

$$\frac{d\phi_i}{dt} = \omega_i - K h_i \sin(\phi_i - \theta_i)$$

Annealed approximation

$$a_{ij} \approx \frac{k_i k_j}{N\langle k \rangle}$$

Global field per link

$$H e^{i\theta} \equiv \frac{h_i}{k_i} e^{i\theta_i} = \frac{1}{N} \sum_{j=1}^{N} \frac{k_j}{\langle k \rangle} e^{i\phi_j}$$

$$\Delta e^{i\theta} \equiv \frac{1}{N} \sum_{j=1}^{N} e^{i\phi_j}$$

$$\frac{d\phi_i}{dt} = \omega_i - K k_i H \sin(\phi_i - \theta)$$

$$\frac{d\phi_i}{dt} = \omega_i - K \Delta \sin(\phi_i - \theta)$$

Oscillators with high k experiences strong coupling field kK.

Self-consistency equation

Random frequency : Uniform dist.

$$H = \frac{KH}{2b\langle k\rangle}\left[\int_{k<b/KH}\frac{\pi}{2}k^2P(k)dk + \int_{k>b/KH} f\left(\frac{b}{kKH}\right)k^2P(k)dk\right]$$

ER random networks

$$\Delta = \frac{\pi K\Delta}{4b} \quad \text{for } |K\Delta| \le b$$

$$= \frac{K\Delta}{\ } f(b/K\Delta) \quad \text{for } |K\Delta| > b$$

^MF^ ≁`MF`

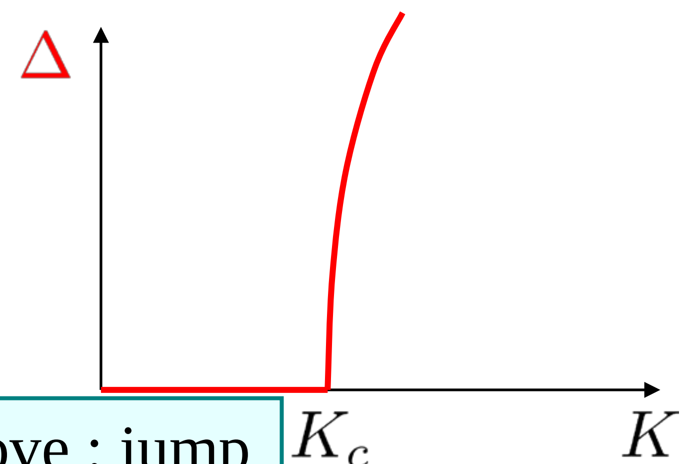
Different from CG result

$$H \approx \frac{\pi K\langle k^2\rangle H}{4b\langle k\rangle} - \frac{1}{\langle k\rangle}\sqrt{\frac{KH}{2b}}P\left(\frac{b}{KH}\right)$$

$$\Delta \sim H \sim \frac{b}{K_c}|\ln(K-K_c)|^{-1}$$

$$K_c = \frac{4b\langle k\rangle}{\pi\langle k^2\rangle}$$

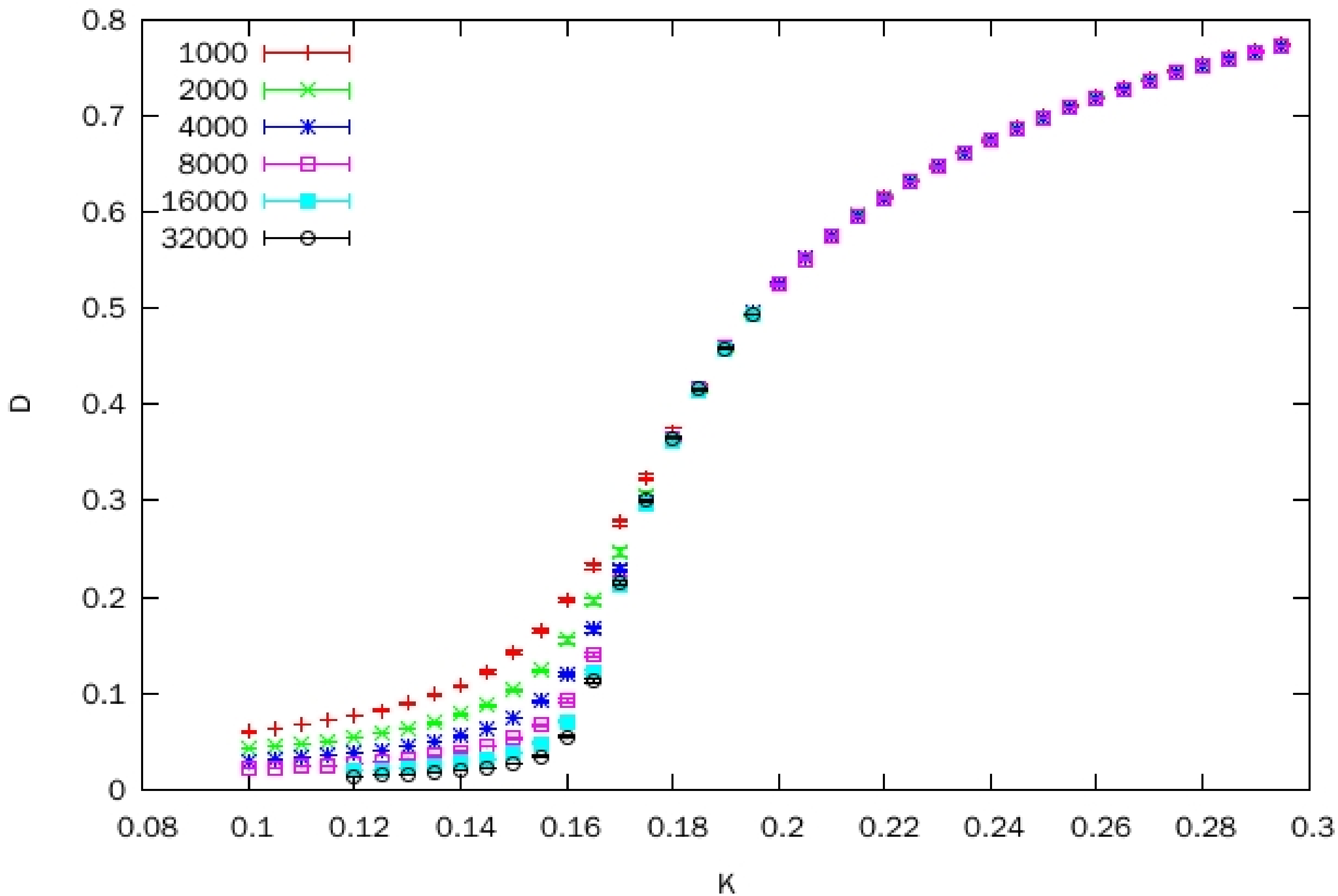
Logarithmic scaling
(very unusual MF)



- degrees bounded from above : jump

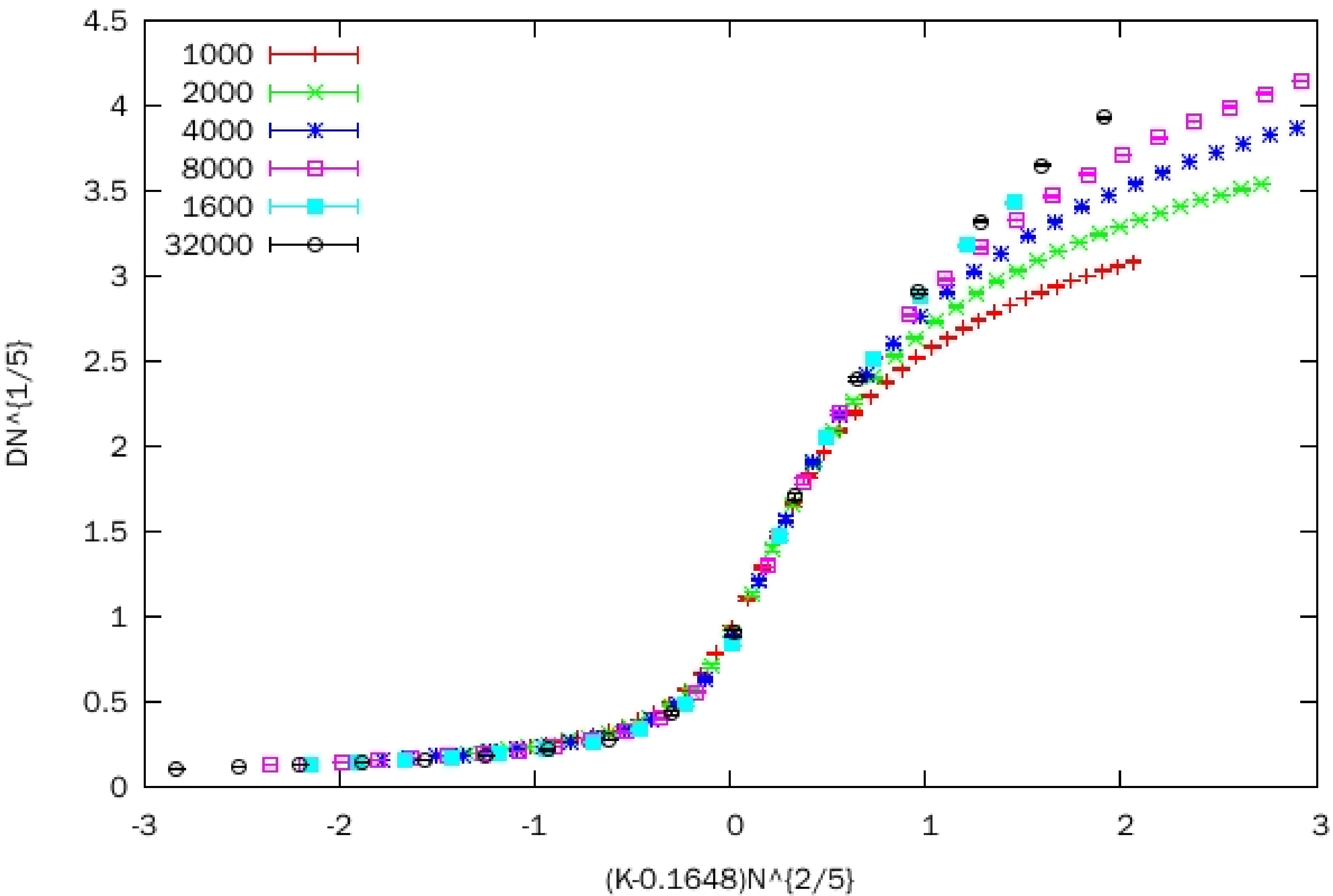
ER random networks : Quenched

Uniform freq. dist.



Scaling collapse

$\beta = 1/2$ and $\bar{\nu} = 5/2$
(same result with unimodal dist. on CG)

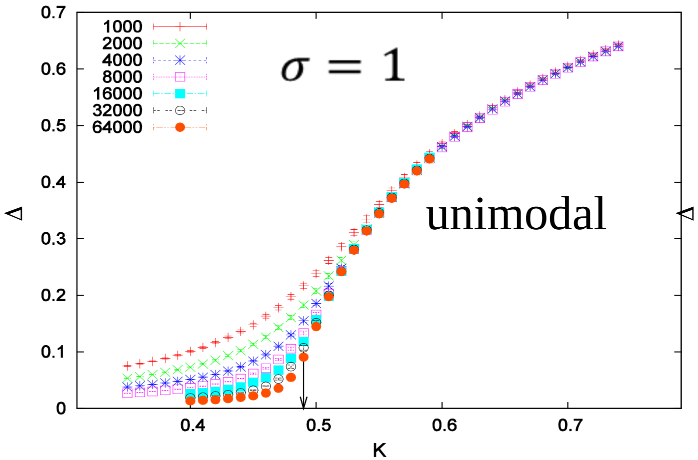


Numerical results on the quenched ER network

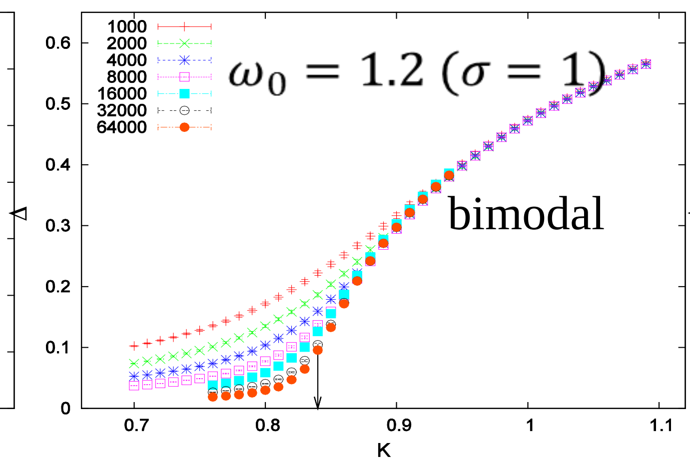
$\langle k \rangle = 4$

$$\dot{\phi}_i = \omega_i - K \sum_{j=1}^{N} a_{ij} \sin(\phi_i - \phi_j) \qquad \Delta = \left\langle \frac{1}{N} \left| \sum_{j=1}^{N} e^{i\phi_j} \right| \right\rangle \sim \sqrt{K - K_c} \text{ regardless of } g(\omega)$$

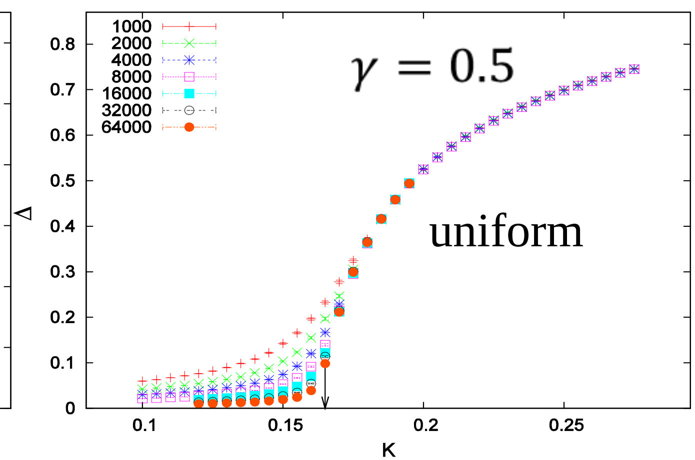
MF: 2^{nd} order

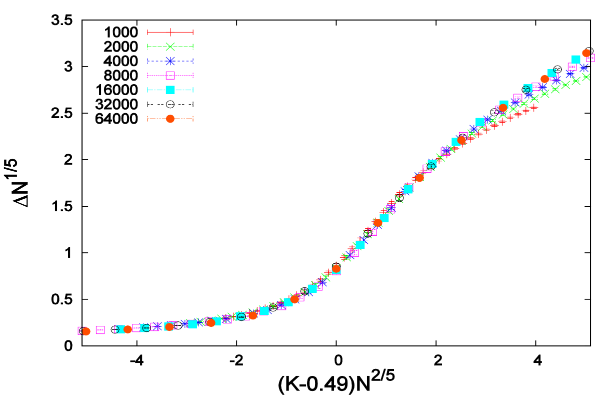


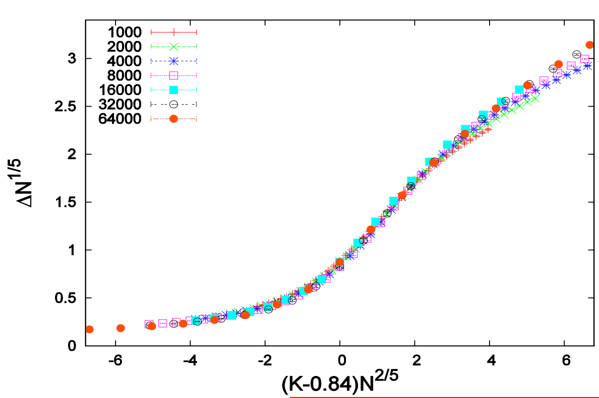
MF: 1^{st} order

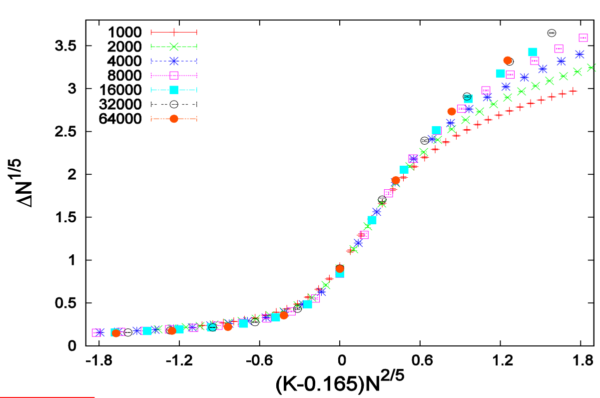


MF: log behavior









FFS: CG exponents $\boxed{\beta = 1/2 \ \ \bar{\nu} = 5/2}$

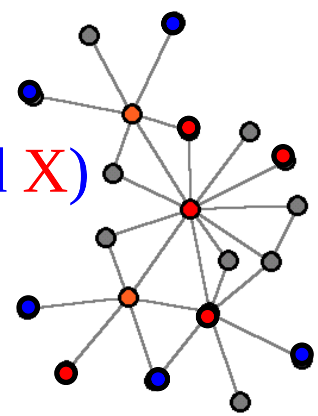
Synchronization with uniform/bimodal freq. distribution

- Results on CG & annealed & quenched ER networks are ALL different !!!

- Results on quenched networks seem identical to the results of the ordinary MF theory with unimodal freq. distribution.

Why?

- Oscillator frequency is effectively modified by the environments through interactions.



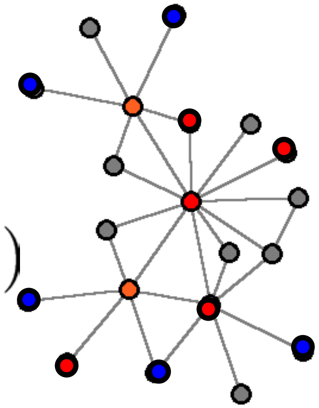
* Additional distribution (noise) may be relevant ! (annealed X)

* Some low-freq. oscillators in hostile environments could not participate in synchronization. . (CG X)

* High-freq. oscillators cannot destabilize all low-freq. oscillators due to limited (quenched) connections. (CG X)

"Effective" frequency distribution before the onset of macroscopic synchronization ??

Effect of additional "additive" noise



$$\boxed{\omega_i^e \equiv \omega_i + \eta_i}$$

$$\boxed{\text{with} \quad \eta_i \sim \frac{1}{k_i}\sum_j a_{ij}\omega_j} \text{(average n.n } \omega\text{)}$$

For simplicity, assume η_i are independent and Gaussian random variable with $\langle \eta^2 \rangle \sim \langle \omega^2 \rangle / \langle k \rangle$.

η_i, η_j are NOT independent, if nodes i, j share the same neighbors.

Effective frequency distribution

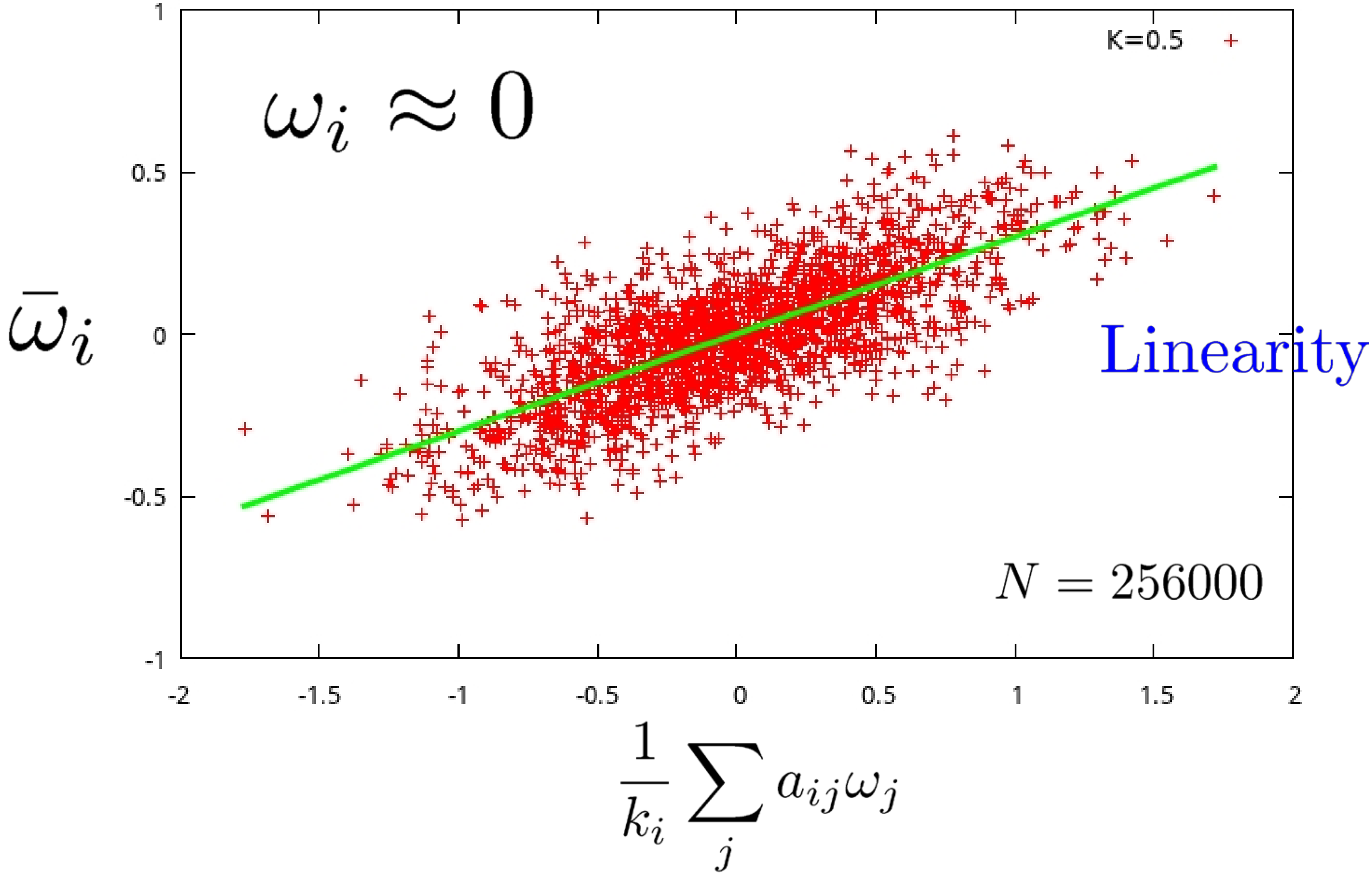
$$\tilde{g}(\omega^e) = \int\int \delta(\omega^e - \omega - \eta) g(\eta) g(\omega) d\omega d\eta \sim \int e^{-\langle k \rangle (\omega^e - \omega)^2/2} g(\omega) d\omega$$

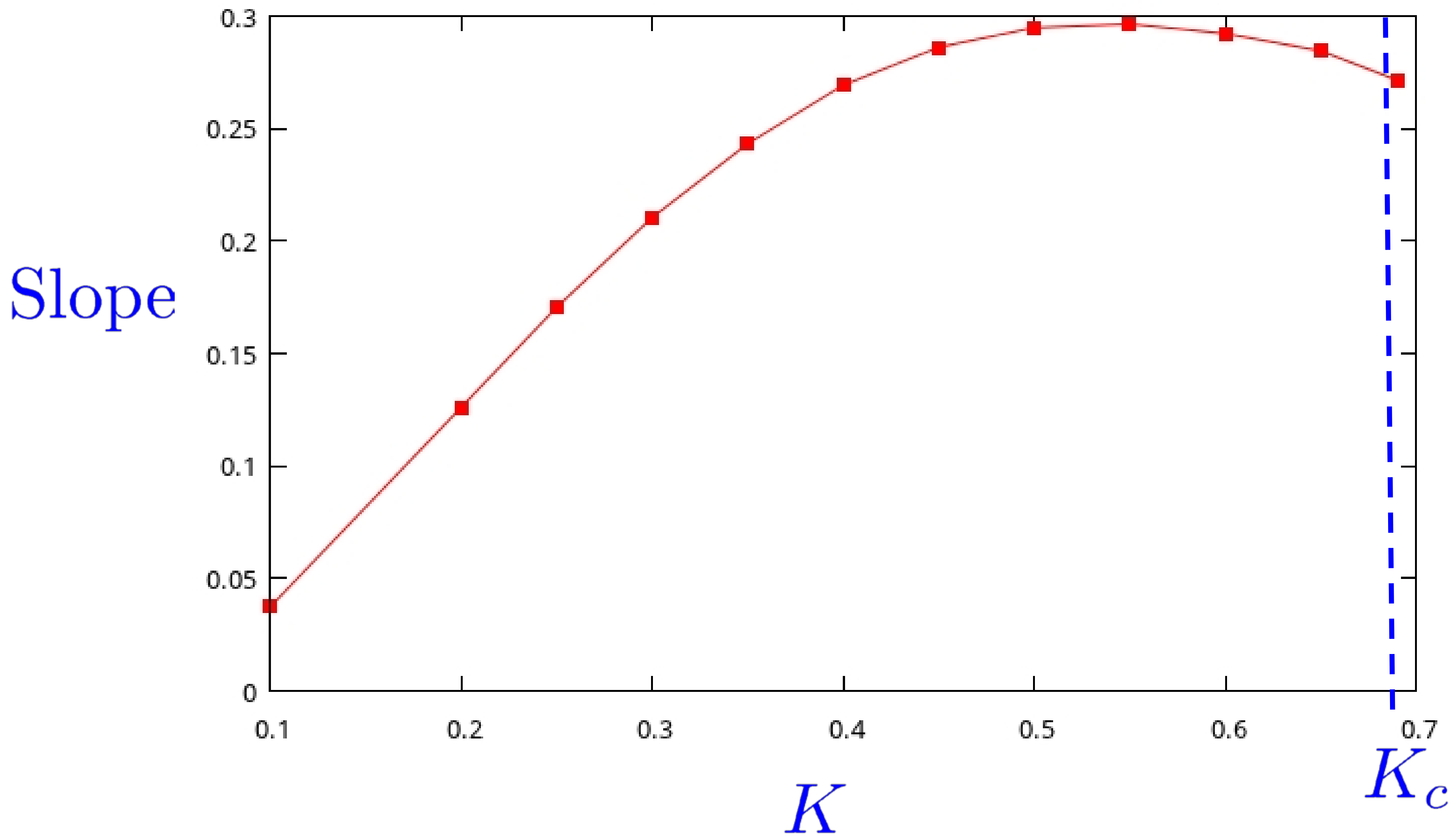
- Additional noise gives additional concaveness near zero frequency !

Possibility for the 2nd order continuous transition even for uniform and bimodal freq. distribution

$$\boxed{\omega_i^e \equiv \omega_i + \eta_i}$$

regular RN with $k = 4$





Slope
0.3
0.25
0.2
0.15
0.1
0.05
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
K
K_c

Remarks

• How much can you convince yourself that you are doing a right calculation (annealed approximation) even when you are dealing with seemingly innocent "uncorrelated" Erdos-Renyi random networks?

• How one can categorize the cases where simple annealed MF calculations are good enough for sparse ER or SF networks ?

• The MF theory on the complete graph may not work else where.

• Which case applies to high-dimensional MF behavior?

• Up to now just case by case. Any systematic understanding possible?

• Still lacks a full story (understanding) even for synchronization prob.