Spectral Analysis on Explosive Percolation

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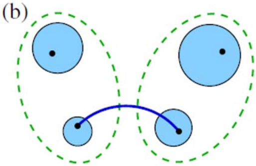
Building a network

- Start with a set of N nodes and establish links/edges one at a time according to some algorithm
- Start with some finite set of nodes and links/edges, and have an algorithm to add a new node to the set

Here we are focusing on what happens to the network when the number of connections between nodes in the network is increased following a suppression principle: growth of all clusters are suppressed.

A simple suppression mechanism

- Suppression of cluster size:
  1. Choose 2 nodes at random, compare the clusters' sizes, select the one in the smaller cluster
  2. Choose a second pair of nodes, do the same as above
  3. Add a link between the two selected nodes



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Emergence of connectivity in a network

Giant component: the cluster in the network with the largest number of nodes

- Random network: number of nodes in the giant component grows smoothly
- Explosive percolation: number of nodes in the giant component grows abruptly

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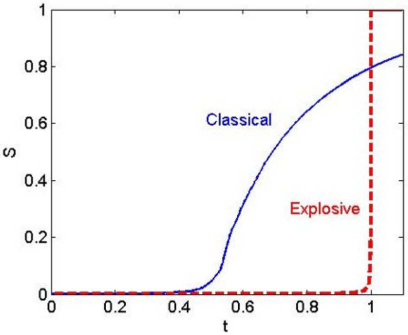
Classical
Explosive
S
t
0
0.2
0.4
0.6
0.8
1

Explosive percolation

- Achlioptas process: formation of large components is suppressed, *sudden emergence of large-scale connectivity*
- Emergence of giant component
  - possible mechanism for growth process of real-world networks (e.g. human protein homology network)
  - increase the possible extent of viral (rather than localized) outbreak? Communication speed? Information transmission?
  - drastic change in macroscopic connectivity by addition of a single link: effect on network dynamics and function? (neural network, social relations, etc.)

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Spectral Analysis

- Input: the maximum eigenvalue of the adjacency matrix governs how the spreading (of information, energy, disease, etc.) on a network
- Study the maximum eigenvalue λ_m of the adjacency matrix
- If λ_m increases sharply $\rightarrow$ enhanced efficiency of spreading
- If λ_m remains small $\rightarrow$ low efficiency of spreading despite the emergence of large-scale connectivity
- Explosive models explored:
    1. Smallest cluster model
    2. Gaussian model

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Smallest Cluster (SC) Model

1. Begin with $N(= 2^n)$ isolated nodes

2. At each step
   - identify the two smallest clusters
   - add an edge between them
3. Phase 1: first $N/2$ steps $\rightarrow N/2$ clusters of size 2

Smallest Cluster (SC) Model

4. Phase 2: next $N/4$ steps $\rightarrow$ $N/4$ clusters of size 4

5. ...
6. Phase $n-1$: 2 components remain, each with size $N/2$

7. Phase n: Step $N-1$, the size of the largest component jumps in value by $N/2$

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Some analysis:

- At phase y:
  - $N/2^y$ edges are added, total degree increases by $N/2^{y-1}$, and

$$\frac{\partial P(k,y)}{\partial y} = \frac{1}{2^{y-1}}\{-P(k,y) + P(k-1,y)\}$$

- Largest degree k_m:
  - probability of node with degree y at phase y is

$$P(y,y) = 1/2^{\eta}, \quad \eta = \sum_{j=0}^{y-1} j$$

  - critical phase:

$$P(y_c,y_c) = \frac{1}{N} \quad \Rightarrow \quad y_c(y_c-1) = 2n$$

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Maximum eigenvalue, SC model

- For fixed N
  - maximum degree k_m of network increases linearly with the phase in the earlier phases
  - probability for k_m to increase beyond phase y_c is very small
- Maximum eigenvalue of the adjacency matrix

$$\lambda_m \propto \sqrt{k_m}$$

- The size of the giant component increases explosively
- Spreading efficiency does not increase explosively

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Results:

- (a) $N = 2^n$ with $n = 13$, Critical Degree:

$$y_c(y_c - 1) = 2(13) \rightarrow y_c \approx 5.6$$

- (b) $N = 2^n$ with $n = 55$, Critical degree:

$$y_c(y_c - 1) = 2(55) \rightarrow y_c \approx 11$$

- Approximation (circles): that of the largest cluster

$$\lambda_m \approx \sqrt{k_m + \frac{\langle k^2 \rangle}{\langle k \rangle^2} \left\{ \langle k^2 \rangle - \frac{k_m^2}{S} \right\}}$$

- Ensemble average of 20 network realizations (triangles)
- SC Model (blue), Modified SC Model (red)

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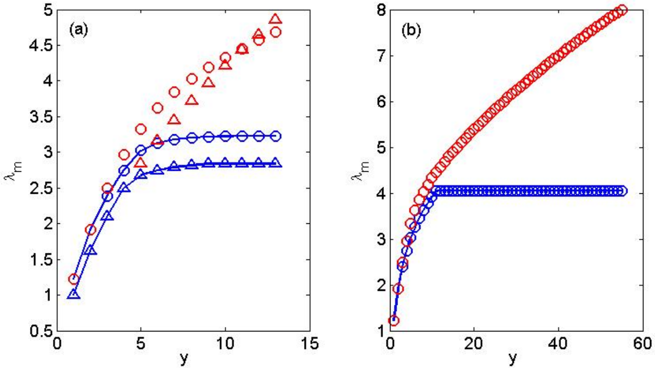
(a) $N = 2^{13}$;   (b) $N = 2^{55}$



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Modified SC model

Slight variation, but still within the model:

- Nodes to be connected in each step are chosen as the largest degree nodes in the two smallest clusters
- Maximum degree increases linearly with the phase
- Larger λ_m and more efficient spreading

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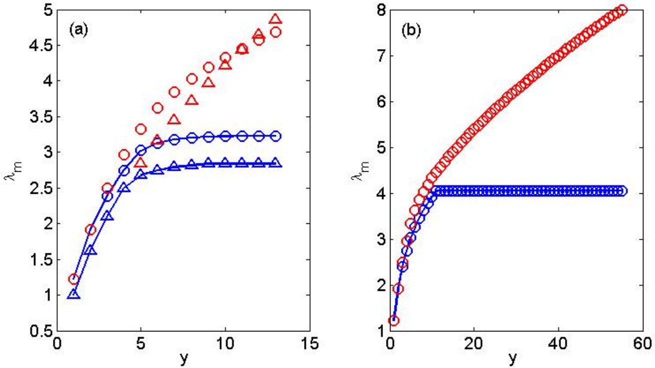
(a) $N = 2^{13}$; (b) $N = 2^{55}$



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Summary

From the simplest explosive models:

1. λ_m does not increase explosively at percolation threshold
2. λ_m is smaller for explosive models compared to random networks
3. Heterogeneous/hub structures can increase spreading efficiency in explosive models
4. Will be interesting to extend to other models
5. Possible tuning when spreading efficiency and large-scale connectivity are not needed concurrently

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