”The Synchronization of Interdependent Networks Undercomes a Phase Transition”

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Prelude

Let me ***Thank the organizers*** *for the kind invitation.*

*Let me A****cknowledge People*** *I am currently collaborating with on this subject (in alphabetic order):*

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Shlomo Havlin (Ramat-Gan)

Javier Martin Hernandez (Technical University - DEFT)

Eugene Stanley (Boston University)

Piet Van Mieghen (TU-DEFT) & Huijuan Wang (TU-DEFT)*

*Let me A****dvertize*** *the EU project* ***MOTIA*** *that supported the work*

Www.motia.eu

*Let me A****dvertize*** *a* ***new hybrid community*** *born in collaboration with Antonio Scala (CNR) for a liaison between*

Critical Infrastructure Protection & Complexity Science

Netonets.org

Motivation: Critical Infrastructures

ENEA
AGENZIA NAZIONALE PER LE NUOVE TECNOLOGIE, L'ENERGIA E LO SVILUPPO ECONOMICO SOSTENIBILE

*An **Infrastructure** is a set of physical components and humans designed to provide a service or a good.*

*A **Critical Infrastructure** is an Infr. providing a basic service or fundamental goods. Among CI's are **Electric System**, Aqueducts, ICT Assets, Fresh **food** distribution chains, **Gas**ducts, Oil Pipelines, **Transports**, Financial networks etc.*

[EUDIR (2008)114 Dec 8-th]

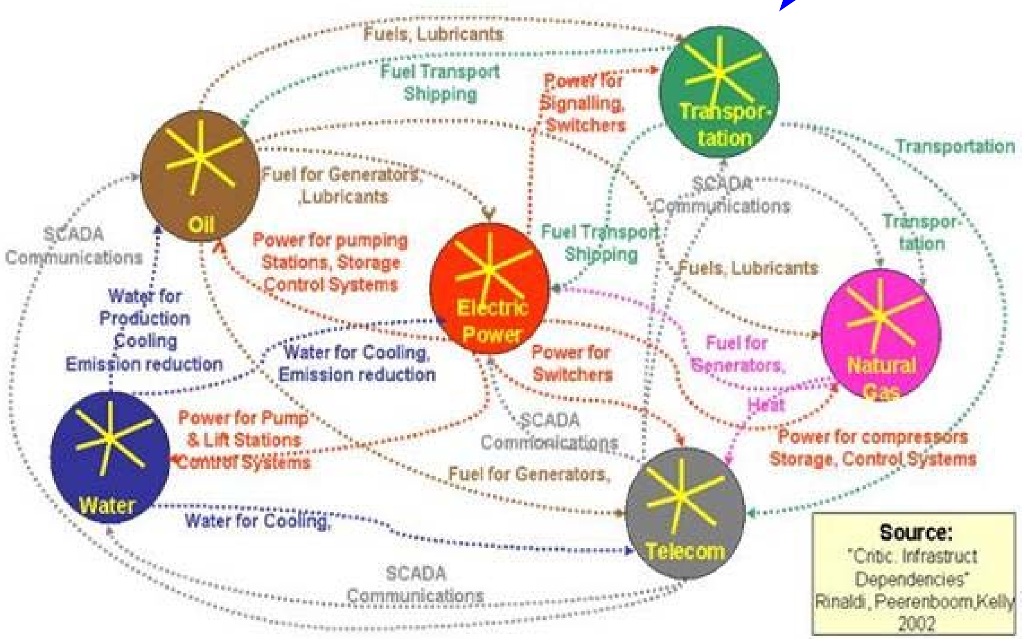
[American Pres. Dir. PDD-63 of May 1998]

__Dependence__ is a condition for which a system (or a component) is required for normal functioning of an other. __Interdependence__ is a mutual dependence among two or more systems or components.

Qualitative Vision

Buldriev et al
Nature (2010)

Interdependent Critical Infrastructures



Quantitative Methods

$\langle \ln Z \rangle$

Complexity Science

(Graph Theory+Stat Mech)

Econophysics

Statistics

Risk &Impact Analysis

$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

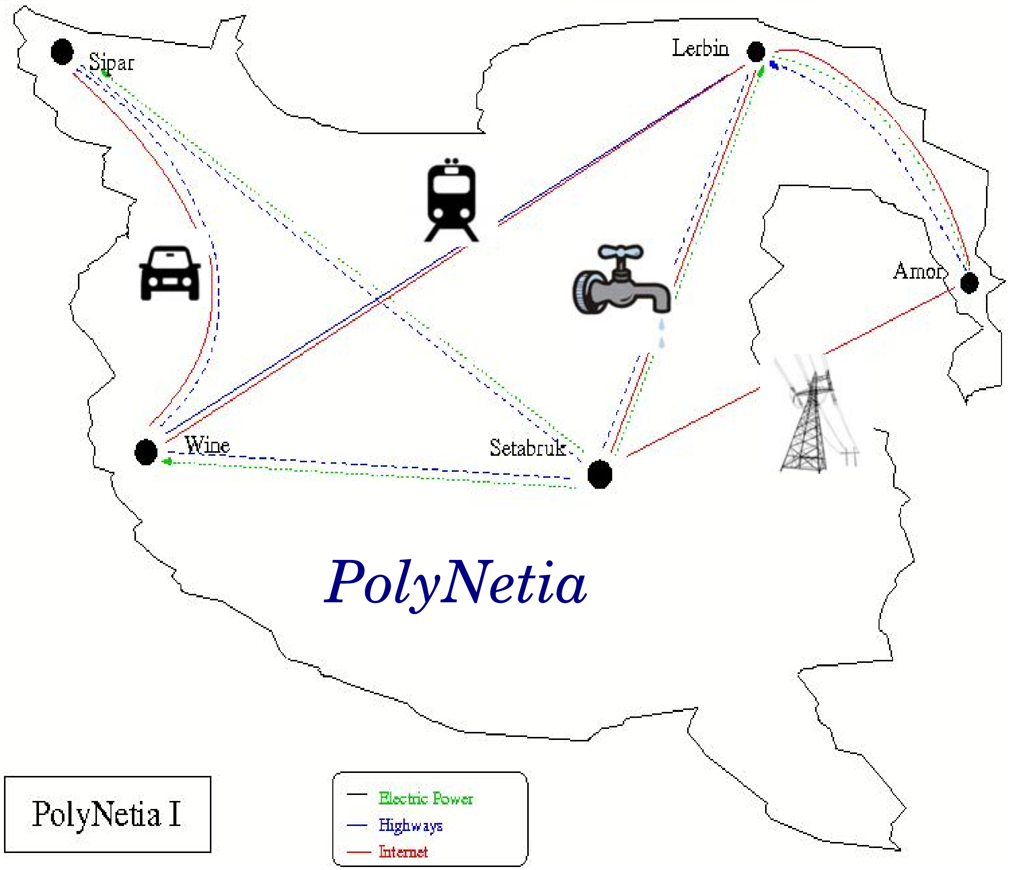
Extended I/O Models

(beyond Leontief)

Modelling & Simulation

Analysis of Signals

NoN – Network of Networks



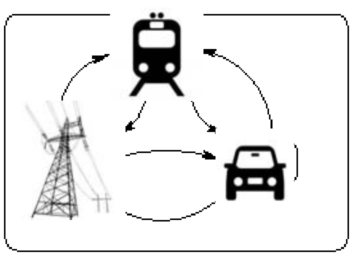
•PolyNetia (PN) is a “virtual country” consisting of different services mutually interacting.

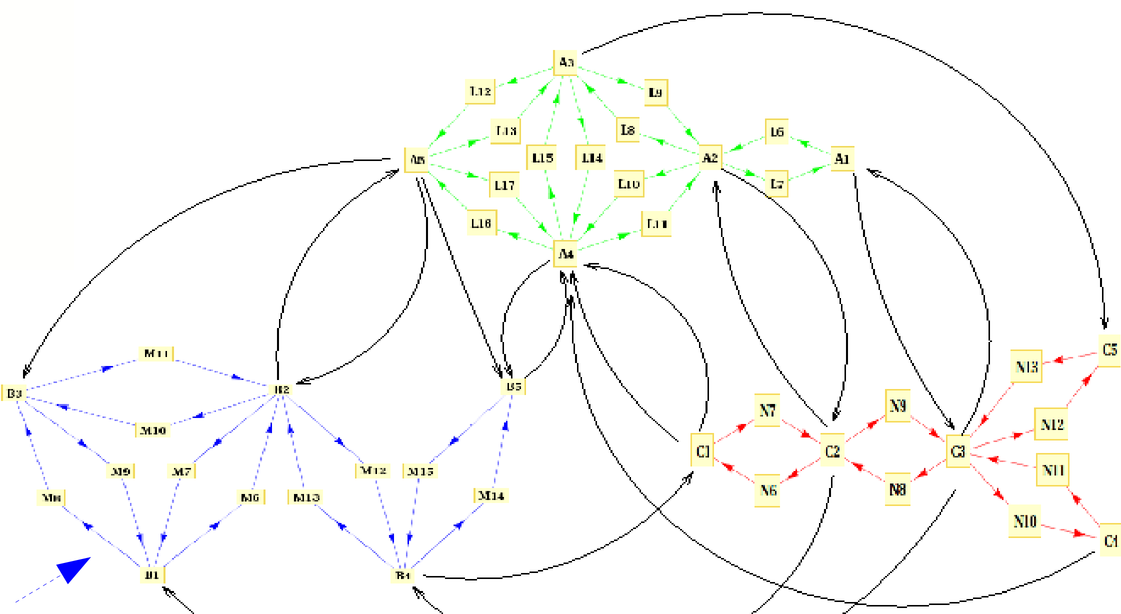
*•PolyNetia is a “developing country” depending on **modeller** aids.*

*•The idea applies to all levels of modeling: from accurate **federation** to **toymodeling**. It may be helpful in many respects:*
Overcoming asset owners privacy.
Simplify reality, thus focussing on crucial points.
Deal with different aspects separately.
Inspect different net structures and governance.

Macro-topology

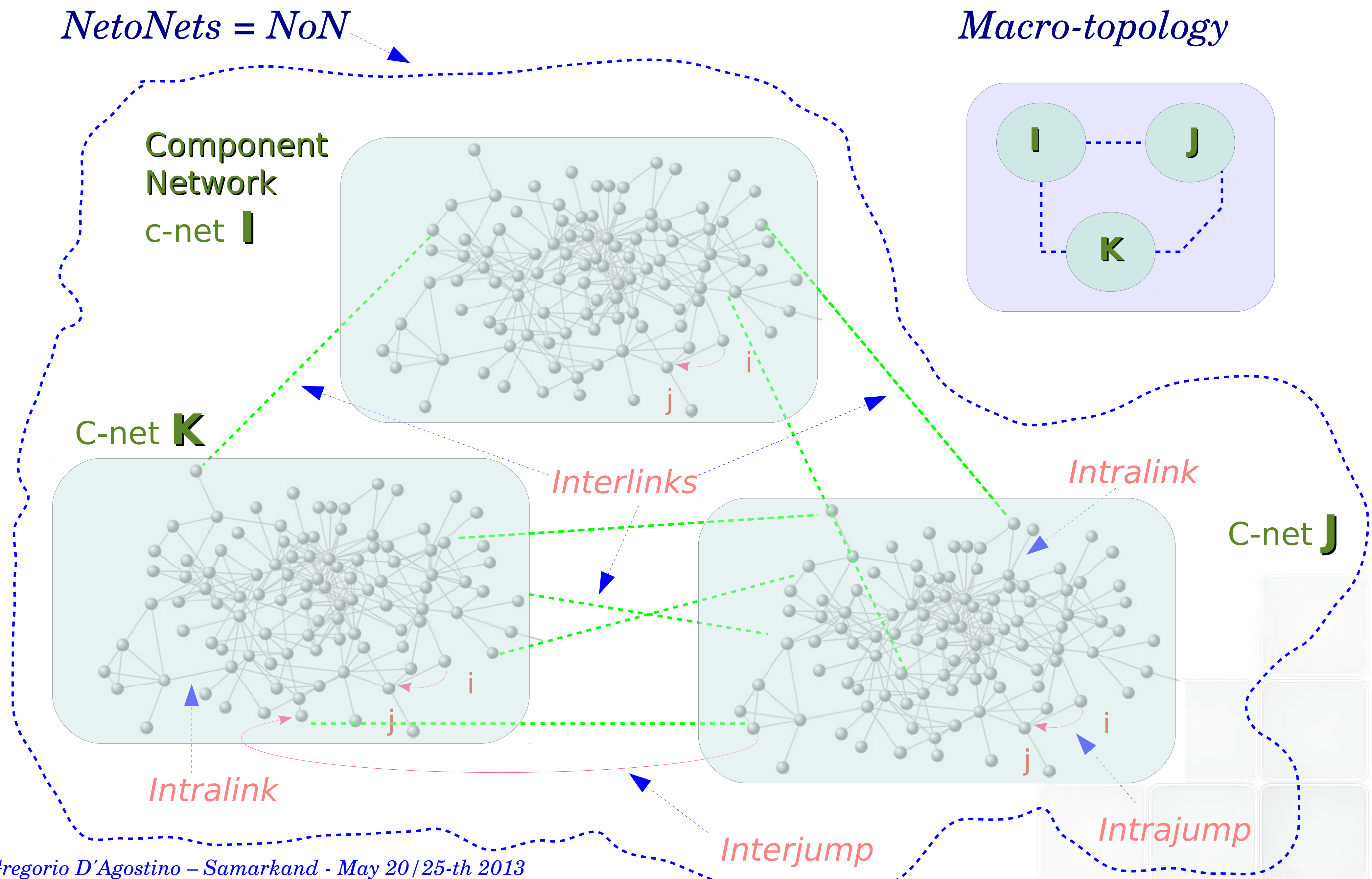
$$A = \begin{pmatrix} 0 \rightarrow A_1 & 1 \rightarrow B_{12} & 1 \rightarrow B_{13} \\ 1 \rightarrow B_{21} & 0 \rightarrow A_2 & 1 \rightarrow B_{23} \\ 1 \rightarrow B_{31} & \rightarrow B_{32} & 0 \rightarrow A_3 \end{pmatrix}$$





G. D'Agostino et al., (2010) LNCS, Springer
“On Modeling of Inter-dependent Network Infrastructures by Extended Leontief Models”

NetoNets Terminology



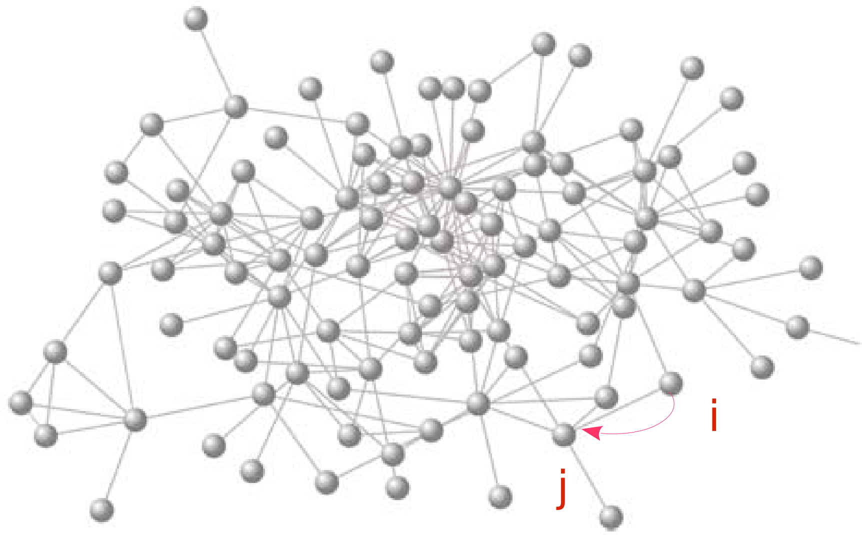
Diffusion Processes on Networks

*In a broad sense, **Diffusion** is a mechanisms that tends to equilibrate a quantity with flows proportional to its discrepancy (gradient). Drunk man walking on a net. The most fashioned application is **"Consensus Dynamics"***

$$\partial_t u_i \sim -\sum_j L_{ij} u_j = -\sum_j A_{ij}(u_j - u_i)$$

$$\sum_j A_{ij}(u_j - u_i) \sim \frac{\partial}{\partial u_i} \sum_j f_{ij}(u_j - u_i)$$

$$P(\xi : i \rightarrow j) \sim A_{ij} \Gamma$$



L is the "Laplacian" of the network

(k_i are node degrees)

$$L_{ij} \stackrel{\text{def}}{=} \delta_{ij} \cdot k_i - A_{ij}$$

When the graph spans a lattice on a continuous manifold and the lattice constant tends to vanish the usual diffusion equations are achieved

$$\partial_t u(x) \sim -\Delta u(x)$$

Macro

$$d\xi \sim \sigma\, dw$$

Micro

Recovery from a deviation

The eigenvalues of the Laplacian represent the inverse of the proper time of the different diffusion modes. The slowest mode corresponds to the first non trivial eigenvalue λ_2.

$$u_i = \sum_{k\,\mathrm{modes}} \xi_i^k c_k \exp(-\lambda_k \cdot t) P^{(k)}(t)$$

When the net is direct and eigenvalues are degenerate $P^{(k)}(t)$ is polynomial of the degeneracy degree.

Controllability *and* ***Synchronizability*** *depend on L* ***spectrum***

The null mode corresponds to a constant (equilibrium) solution.

All deviation from equilibrium are decomposed on the k-modes which "proper times" are the inverse of the L's eigenvalues:

$$\tau_k \overset{\mathrm{def}}{=} \lambda_k^{-1}$$

The slowest of the modes corresponds to the first non trivial eigenvalue. $\tau_{max} = \lambda_2^{-1}$

Bi-Partition Min-Cut

*A **cut** of a connected graph is set of links which removal causes bipartition. The **cut size** is the number of such links.*

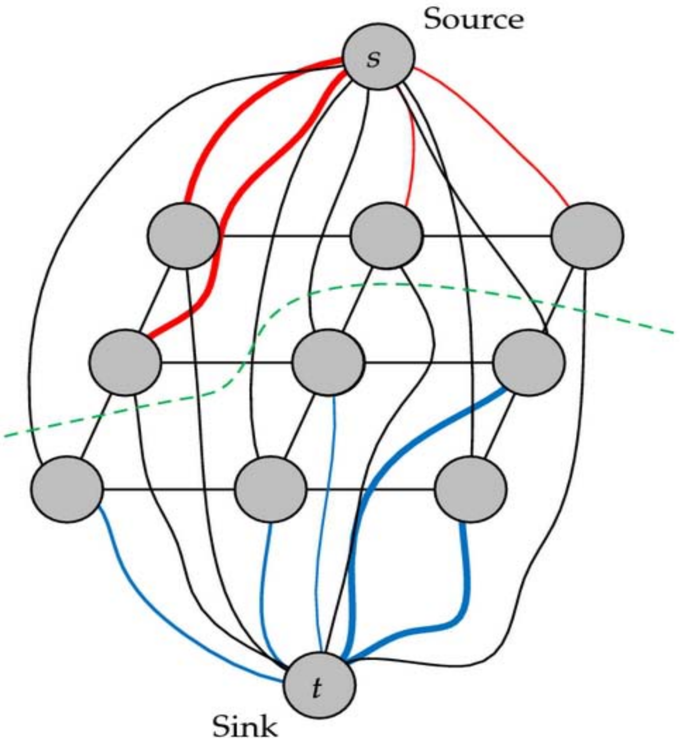
*A **min cut** is a cut of minimum size that separates two given nodes.*

A bipartition can be defined by labeling by a +1 or -1 value the nodes belonging to the two parts.

Ω_+ *is the set of nodes in the largest partition*
Σ *is the set of links in the cut (between nodes of different partitions)*

***Communities** may be found through this means.*

12 links
Min cut



The Algebraic Cut

The algebraic cut is the solution of an optimization problem:

"find the cut of minimum size that results in two equal halves"

Each node is labeled by unitary integers u=1 or -1

The **object function** *H:*

$$H(u) \stackrel{\text{def}}{=} \sum_{ij} A_{ij} \left(\frac{u_i - u_j}{2} \right)^2 = \frac{1}{4} u^T L u. \qquad \sum_i u_i = 0.$$

Relaxing the integer unitary constraint for the x's, while keeping their sum constant, one obtains an algebraic solution of the problem with constrains:

$$u_i \in \mathbb{R} \qquad \sum_i u_i^2 = 1 \qquad \sum_i u_i = 0.$$

The optimum (minimum) of the object function is the lowest eigenvalue of the Laplacian λ_2, while the optimum configuration is given by its eigenvalue ξ:

$$H(u)_{min} = \lambda_2 = \frac{1}{4} ((\xi^2)^T L \xi^2). \qquad \sum_i \xi_i^2 \cdot 1 = 0.$$

Models for Network of Networks

A model network is a set of networks created according to either deterministic or stochastic rules.

*Similarly a model for interdependent networks "**Model NoN**" consists of **three** basic entities:*

*- A **model network** for each component network (→ L_i or A_i in our case)*

*- A **Macroscopic Topology** defining networks dependencies*

*- A **Linkage Strategy** or "linkage rule" (→ B_{ij} in our case)*

BA, WS, 2-D Lattices, and RR will be employed as model networks

***Homologous** Linkage Strategy HLS and*

***Random** Linkage Strategy HLS*

will be mostly employed

NetoNets - Notation for Laplacian

$$L \stackrel{\text{def}}{=} L_0 + L_I$$

$$(d_I)_{ij} = \delta_{ij} \sum_{J=1}^{M} A_{IJ}^{macro} \sum_{k=1}^{N} (B_{IJ})_{ik}$$

$$(L_I)_{ij} = \delta_{ij} \sum_{k=1}^{N} (A_I)_{ik} - (A_I)_{ij}$$

$$\begin{pmatrix} L_1+d_1 & -B_{12} & \dots & -B_{1M} \\ -B_{21} & L_2+d_2 & \dots & -B_{2M} \\ \dots & \dots & \dots & \dots \\ -B_{M1} & -B_{M2} & \dots & L_M+d_M \end{pmatrix} = \begin{pmatrix} L_1 & 0 & \dots & 0 \\ 0 & L_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & L_M \end{pmatrix} + \begin{pmatrix} d_1 & -B_{12} & \dots & -B_{1M} \\ -B_{21} & d_2 & \dots & -B_{2M} \\ \dots & \dots & \dots & \dots \\ -B_{M1} & -B_{M2} & \dots & d_M \end{pmatrix}$$

*The **cNet model networks** tell us the probability distribution of the L_I's*

*The **macrotopology** (A^{macro}) tells us which of the B_{IJ}'s does not vanish*

*The **linkage strategy** tells us the probability distribution of each B_{IJ}*

*All the former three issues define the **NetoNets Model***

A Hierarchy of properties

Exact or Sure *results: expressions (equalities or inequalities) valid for all NoN's of a given model (demonstrated algebraically).*

Almost Sure *results: expressions that take place with Probability One (typically achieved in the thermodynamical limit(s) N,M → Inf).*

Average *and **Stochastic** properties: expressions resulting by averaging over the sample of the given model NoN or being true in a given interval of confidence.*

Approximated *results: achieved by both algebraic and stochastic approximations: Mean-field, simplified Hypothesis, Perturbative results.*

Recent results for the case M=2 have been achieved by Gomez et al
Physical Review Letters, 110, 028701 (2013)

We have provided some further exact inequality and extensively treated the case M=2 - J Martinez et al. arXiv:1304.4731

Independent Linkage Strategy

It is a stochastic linkage strategy where all interlinks are independent and have given a probability to occur:

$$P\left(B_{ij}^{IJ}=1\right)=A^{IJ}\beta_{ij}^{IJ}$$

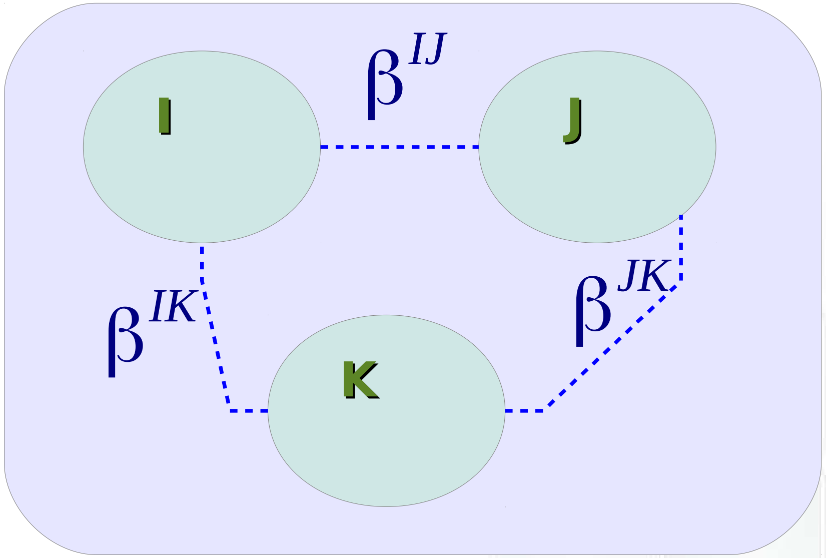
$$P\left(B_{ij}^{IJ}=1;B_{kl}^{KL}=1\right)=P\left(B_{ij}^{IJ}=1\right)P\left(B_{kl}^{KL}=1\right).$$

Where A^{IJ} is the adjacency matrix of the macro topology.

*When the probability function is the same for all pair of interdependent networks the **strategy** is named **uniform**.*

$$P\left(B_{ij}^{IJ}=1\right)=A^{IJ}\beta_{ij}$$

M=3 example



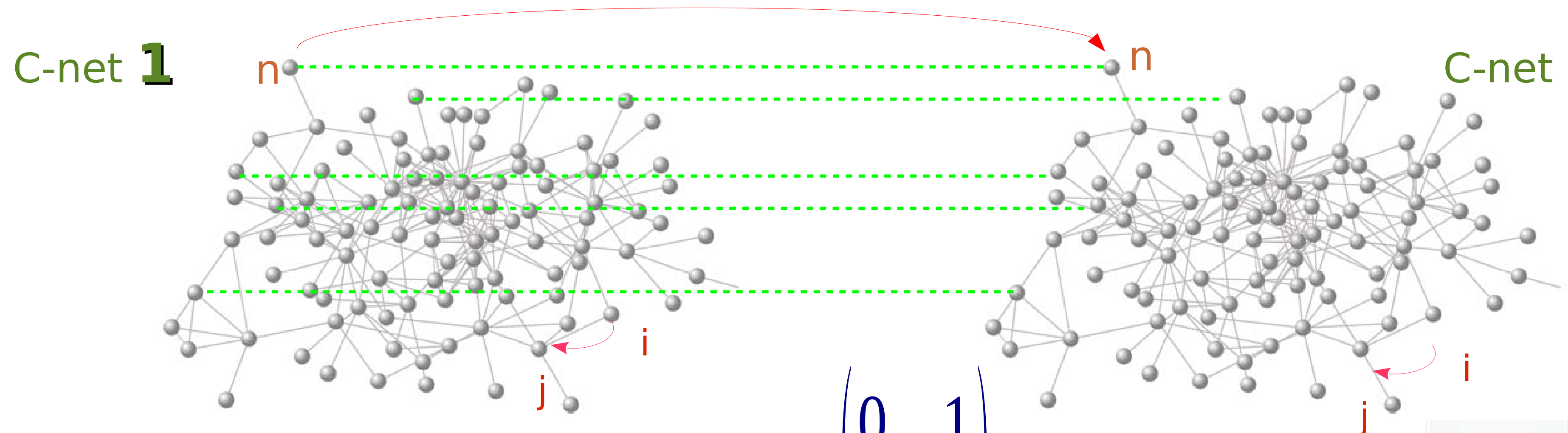
Homologous Linkage Strategy

It is a stochastic linkage strategy where interlinks are allowed only for Nodes having the same index, such links are randomly selected:

$$P\left(B_{ij}^{IJ}=1\right)=A^{IJ}\delta_{ij}\alpha=\delta_{ij}\frac{l}{N}$$

The Group of Permutations allows different definitions of homologous nodes.

Where A^{IJ} is the adjacency matrix of the macro topology; l is the number of links; N is the number of nodes per network and α is the average number of interlinks per node. $\alpha=\left[\frac{l}{N}\right]$

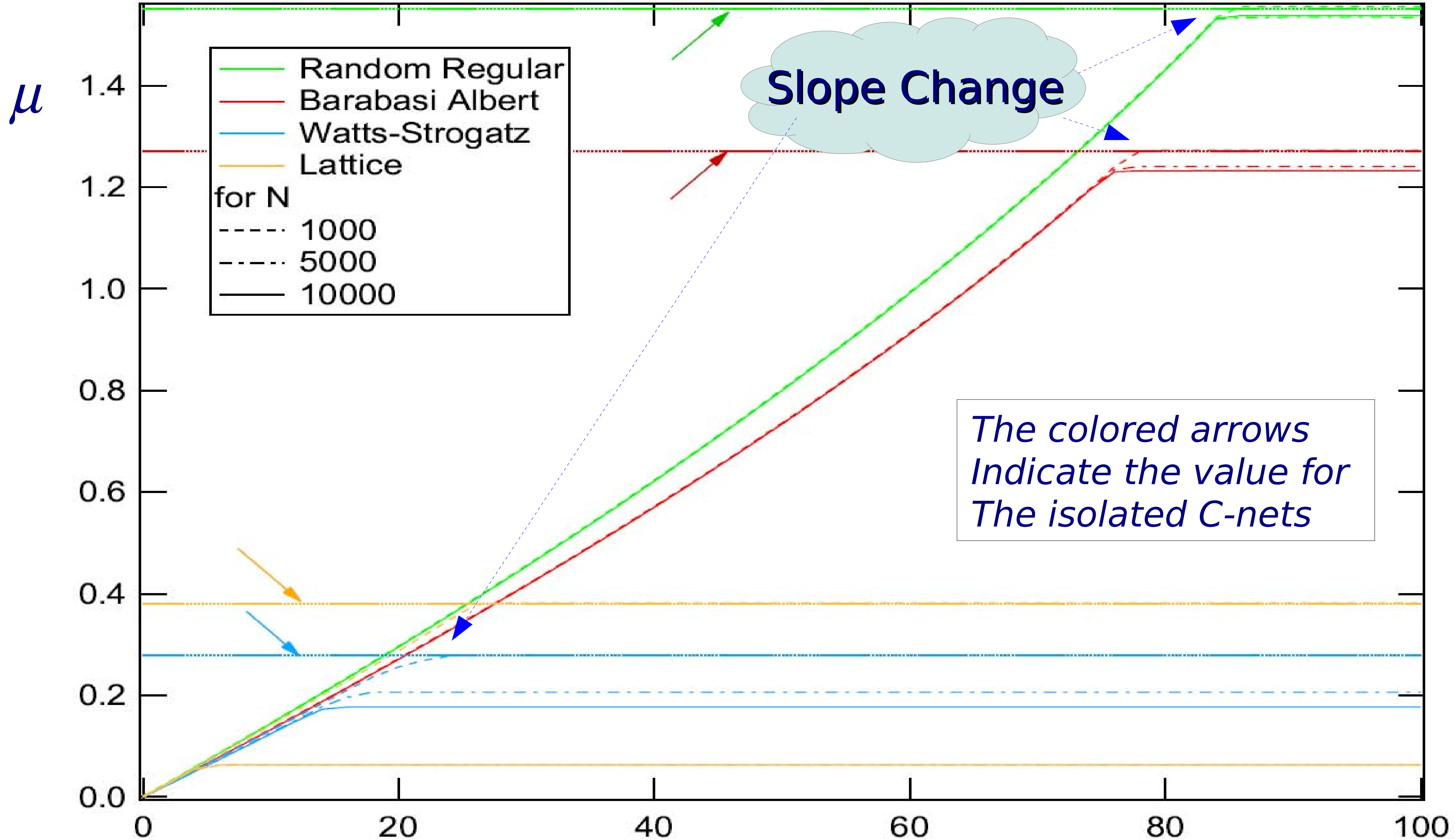


$$A_{macro}=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$$

M=2 example

Numerical Sampling NoN's: μ

M=2 Homologous Linkage strategy – 100 confs each – ***Identical Cnets L_1=L_2***

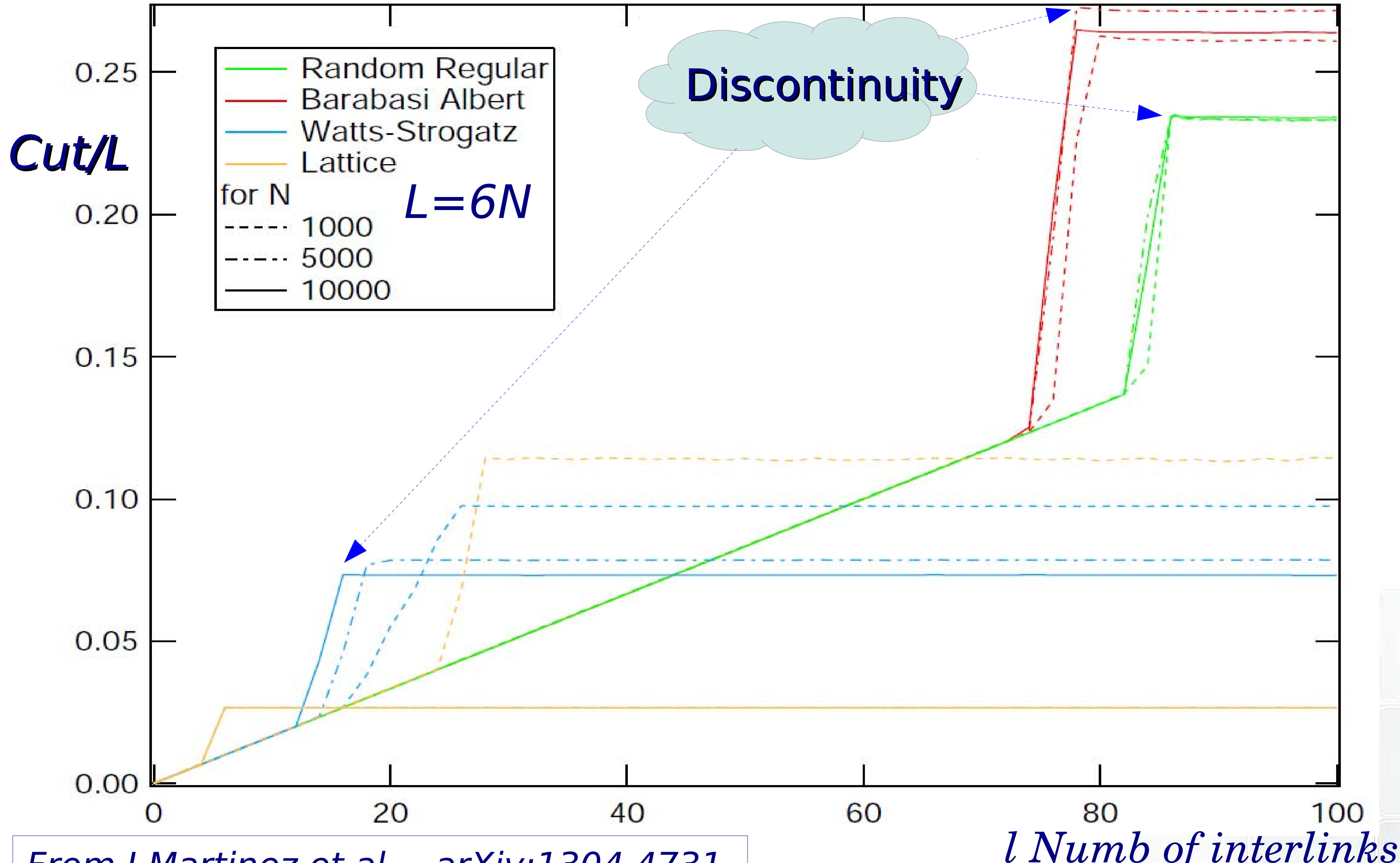


l Numb of interlinks

From J Martinez et al. arXiv:1304.4731

Numerically Sampling: Algebraic Cut

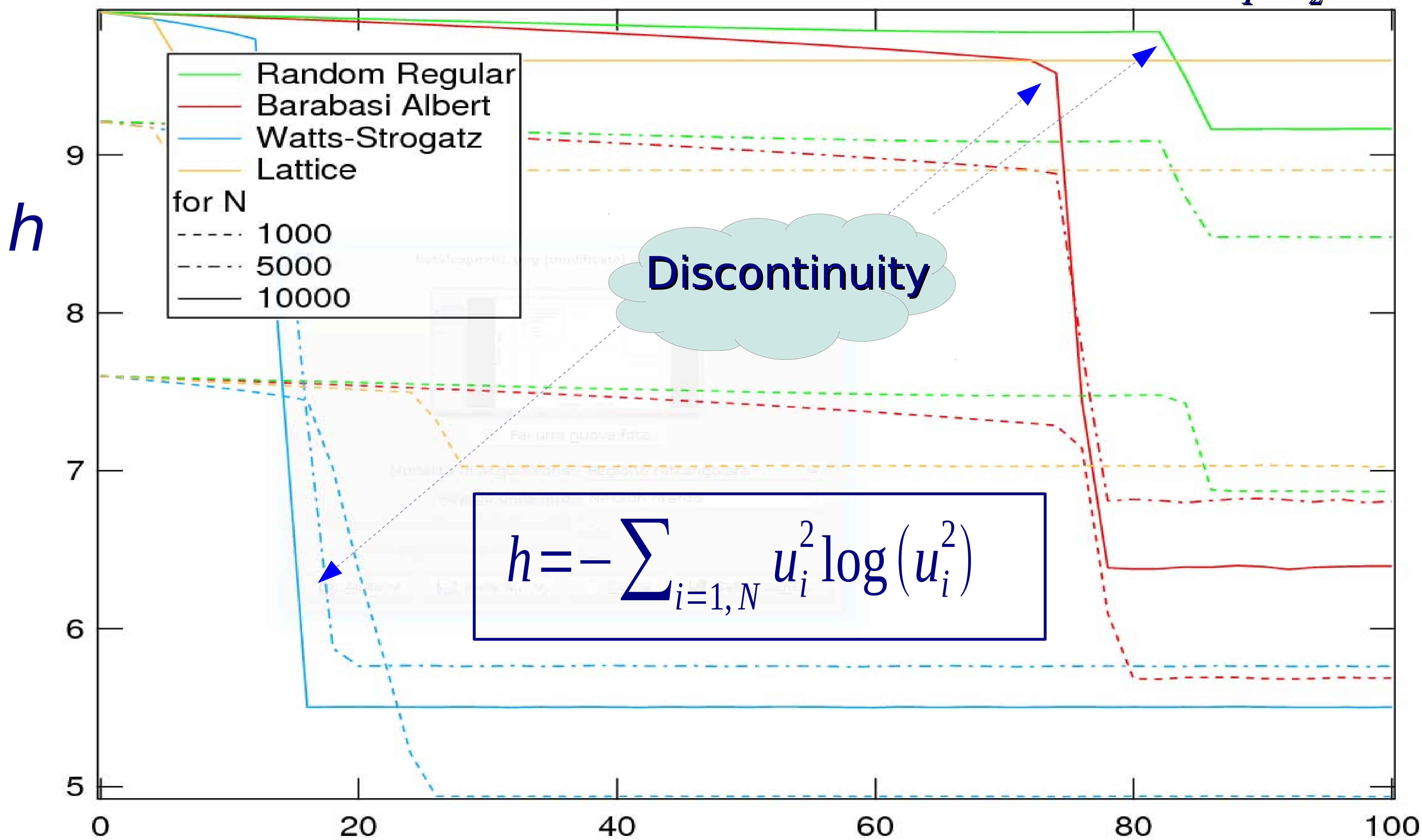
M=2 Homologous Linkage – 100 confs each – ***Identical Cnets $L_1=L_2$***



From J Martinez et al. arXiv:1304.4731

Numerical Sampling: Entropy

M=2 Homologous Linkage – 100 confs each – ***Identical Cnets $L_1=L_2$***



$$h=-\sum_{i=1,N} u_i^2 \log\left(u_i^2\right)$$

l Numb of interlinks

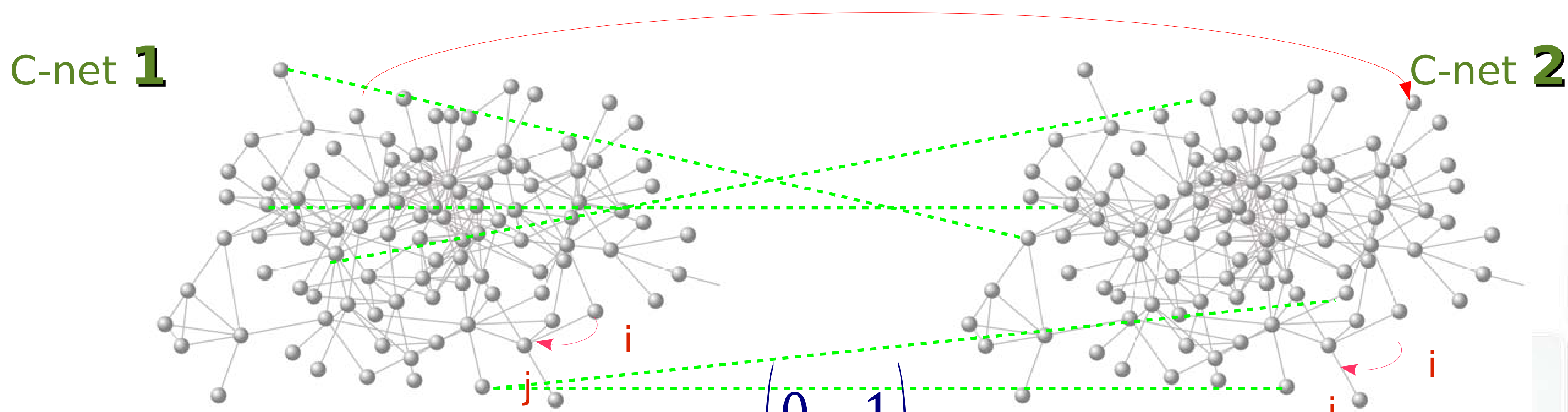
From J Martinez et al. arXiv:1304.4731

Random Linkage Strategy

It is a stochastic linkage strategy where all interlinks are allowed and their probability to occur is uniform.

$$P\left(B_{ij}^{IJ}=1\right)=A^{IJ}\beta=A^{IJ}\frac{l}{N^2}=A^{IJ}\frac{\alpha}{N} \qquad \alpha=\left[\frac{l}{N}\right]$$

Where A^{IJ} is the adjacency matrix of the macro topology; l is the number of links; N is the number of nodes per network and α is the average number of interlinks per node.

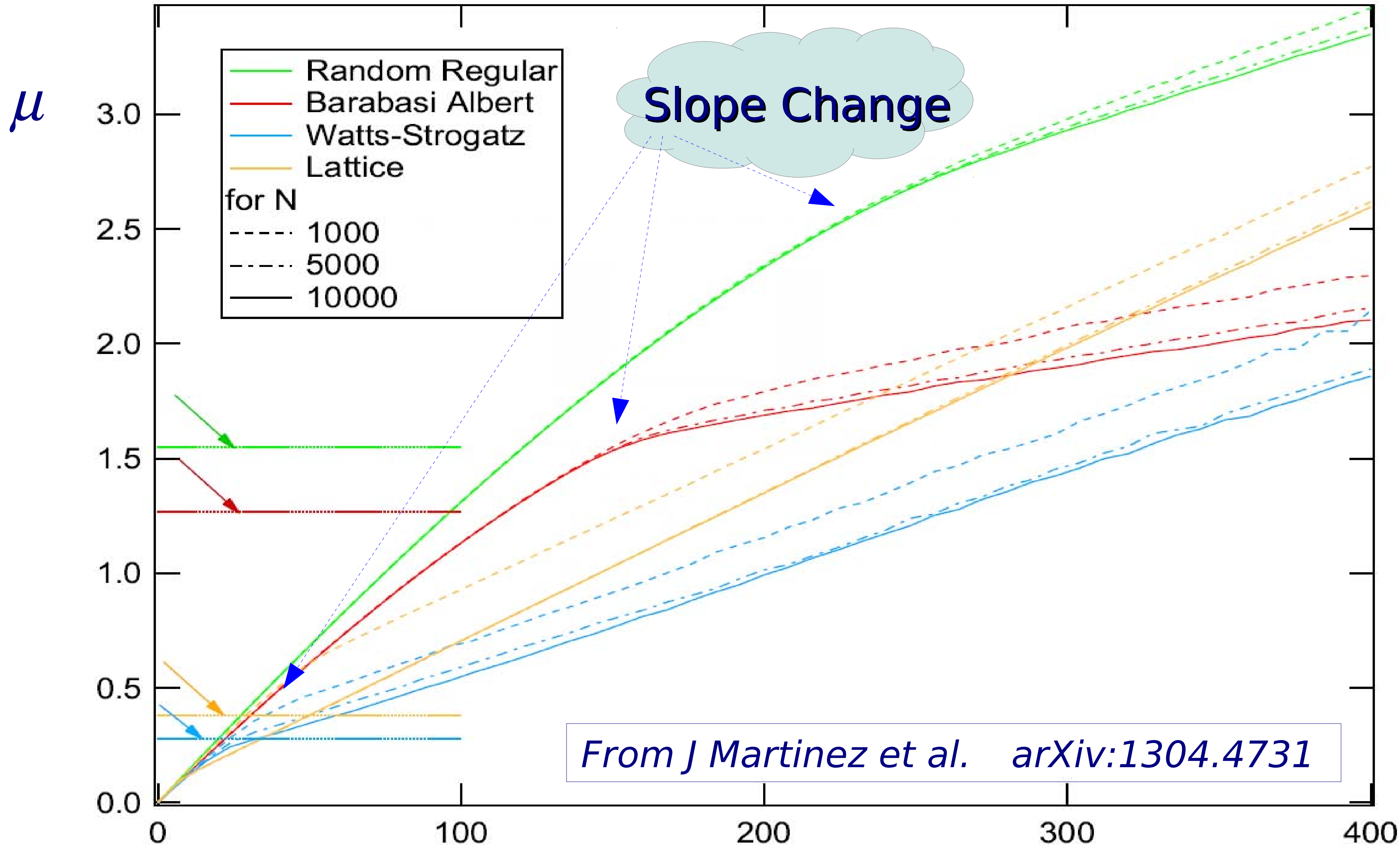


$$A=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

M=2 example

Numerically Sampling NoN's - RLS

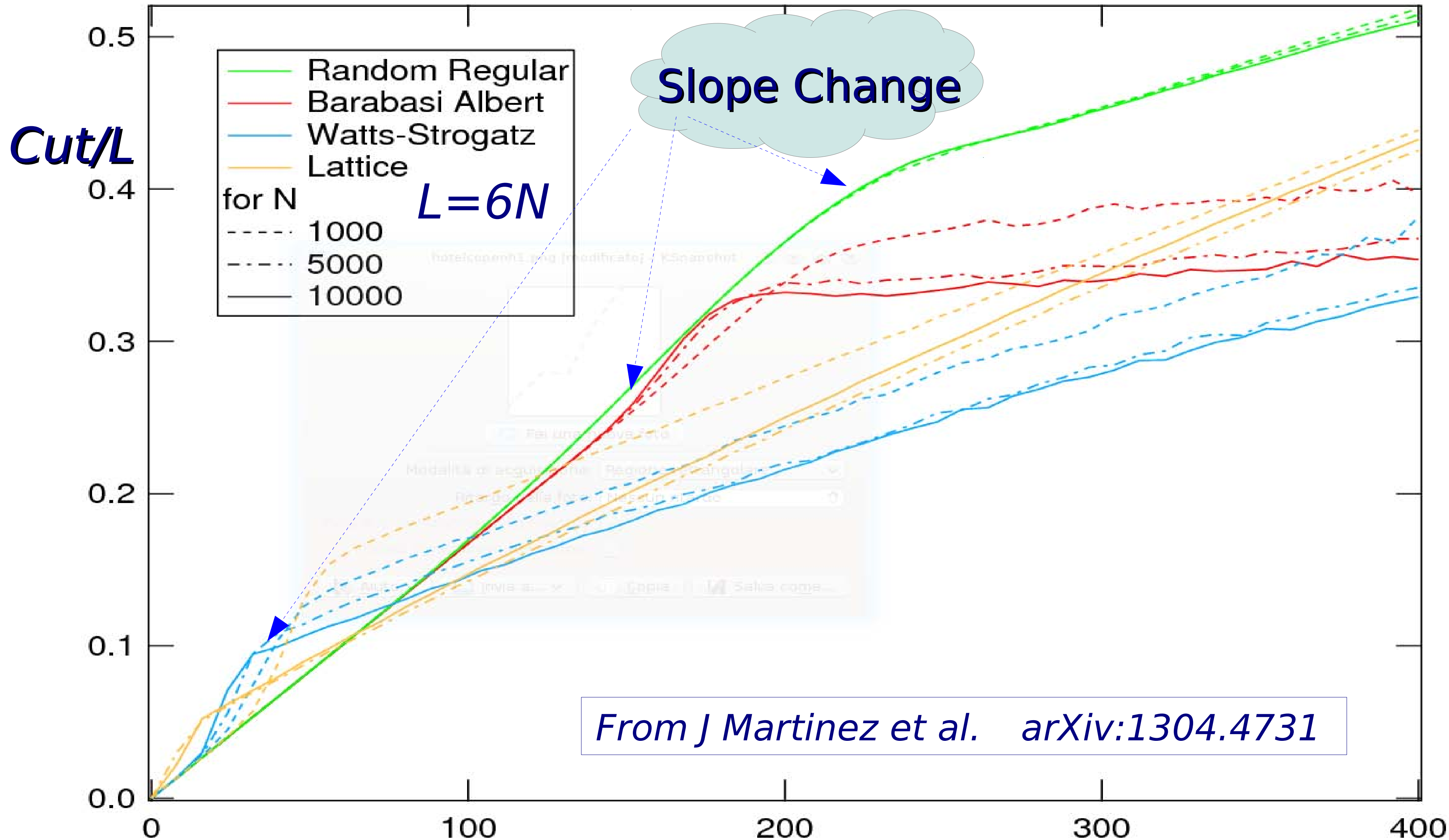
*M=2 **Random Linkage** Strategy – 100 configurations each*



Percentage of interlinks per node $\alpha = l/N$

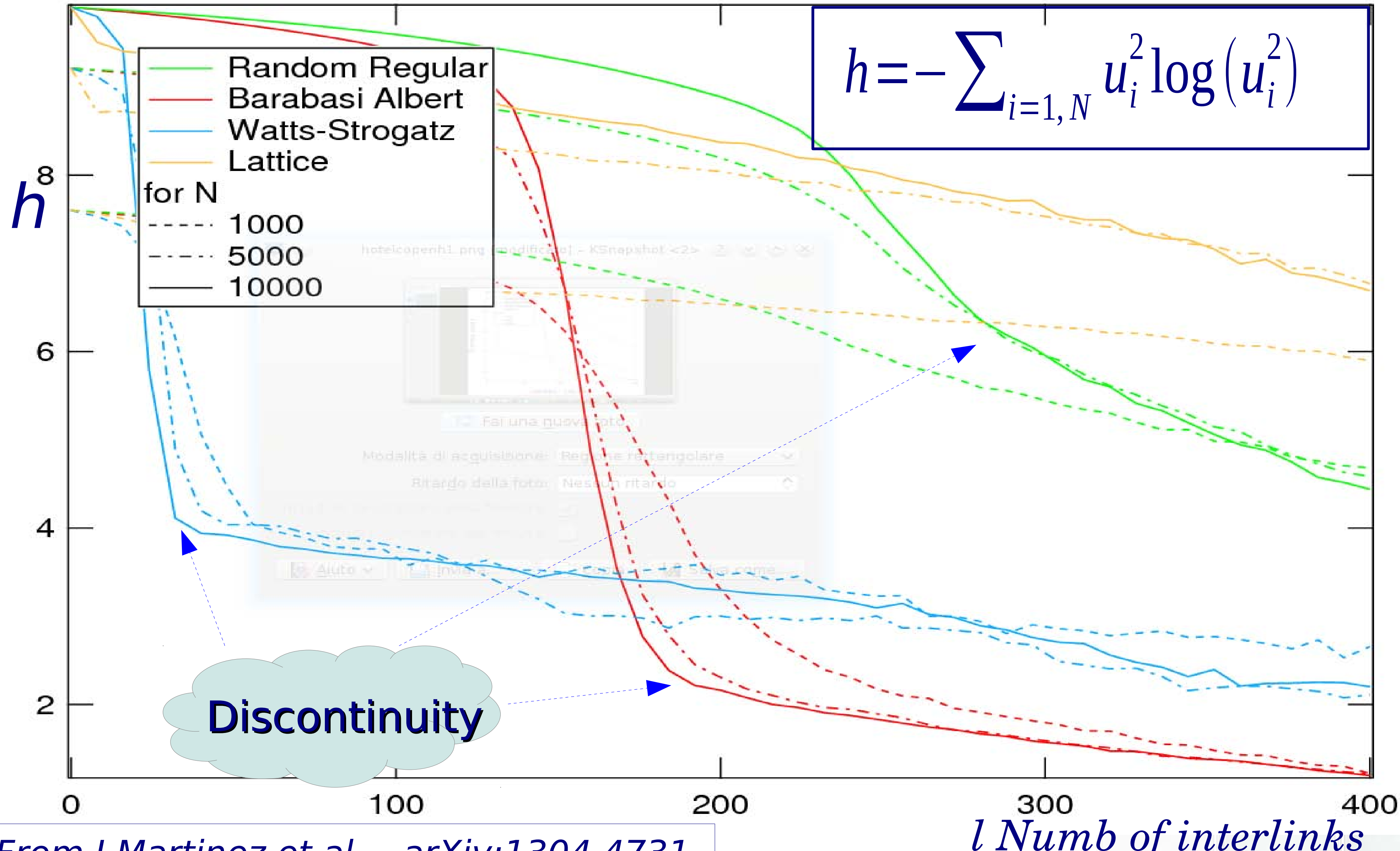
Numerical Sampling: Fiedler Cut

M=2 Random Linkage Strategy – 100 configurations each



Numerical Sampling: Entropy

M=2 Random Linkage Strategy – 100 confs each – ***Identical Cnets $L_1=L_2$***



$$h = -\sum_{i=1,N} u_i^2 \log\left(u_i^2\right)$$

From J Martinez et al. arXiv:1304.4731

Explaining the Phase Transition

There is a critical value of the fraction of interlinks beyond which either the algebraic connectivity either stop growing or reduces its slope.
In the same point the entropy associated with the Fiedler partition experiences a drastic damp. The size of Fiedler cut also experiences a slope change.
One can resort to different means to explain the origin of the phase transition:

- ***Mean-field*** *approach* $E\left[\mu(L)\right] \sim \mu\left(E\left[L\right]\right)$

$$L = \begin{pmatrix} L_1 + I\alpha & -I\alpha \\ -I\alpha & L_1 + I\alpha \end{pmatrix}$$

HLS

$$J = \begin{pmatrix} 1 & 1 & \dots & 1 \\ 1 & 1 & \dots & 1 \\ \dots & \dots & \dots & \dots \\ 1 & 1 & \dots & 1 \end{pmatrix} \qquad L = \begin{pmatrix} L_1 + \beta N & -\beta J \\ -\beta J & L_1 + \beta N \end{pmatrix}$$

RLS

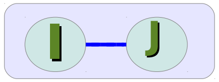
Perturbation theory

$$\mu = \mu(\alpha) = \mu^{(0)} + \mu^{(1)}\alpha + \mu^{(2)}\alpha^2 + \dots$$

Exact + Statistical Results

Mean-Field for HLS

The case M=2 is the simplest. The Macrotopology is trivial



$$A=\begin{pmatrix} 0\to A_1 & 1\to B_{12} \\ 1\to B_{21} & 0\to A_2 \end{pmatrix} \qquad L_{macro}=\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \qquad L=\begin{pmatrix} L_1+d_1 & -B_{12} \\ -B_{21} & L_2+d_2 \end{pmatrix}$$

$$L=\begin{pmatrix} L_1+I\alpha & -I\alpha \\ -I\alpha & L_1+I\alpha \end{pmatrix}=L_0+\alpha L_I$$

L_0 and L_I commute → common set of eigenvectors

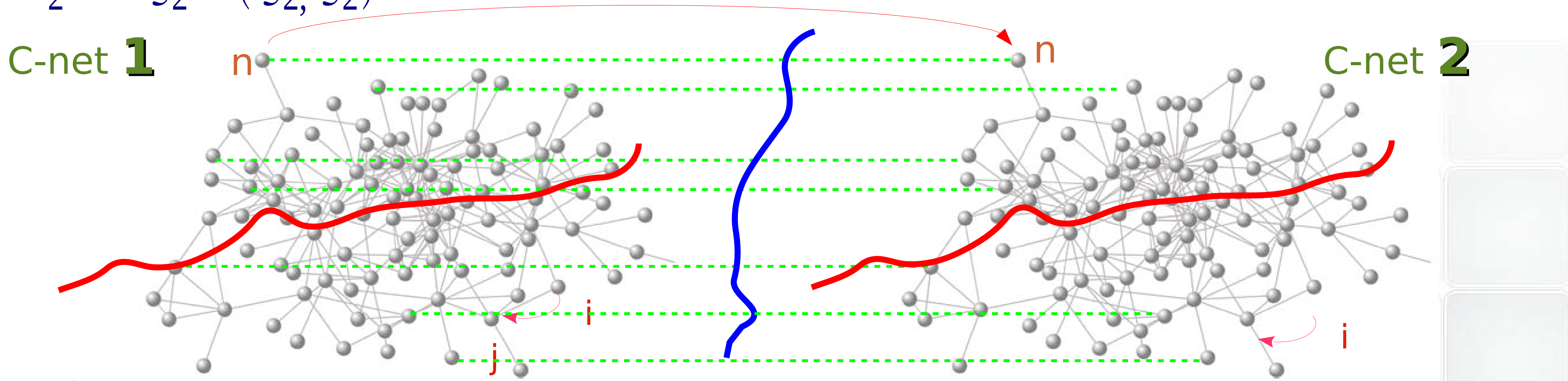
$$v_i^I=\Xi^I\xi_i \qquad L_1\xi_i=\lambda_i \qquad \lambda_1=0,\lambda_2=\mu(C-net_1),\ldots$$

$$L_{macro}\Xi_I=\Lambda_I \qquad \Lambda_1=0,\Lambda_2=\mu(Macro\,Net),\ldots$$

$$\mu_i^I=\Lambda^I+\lambda_i \qquad \mu_1^1=0,\mu_1^2=\Lambda_2\alpha,\mu_2^1=\lambda_2,\ldots$$

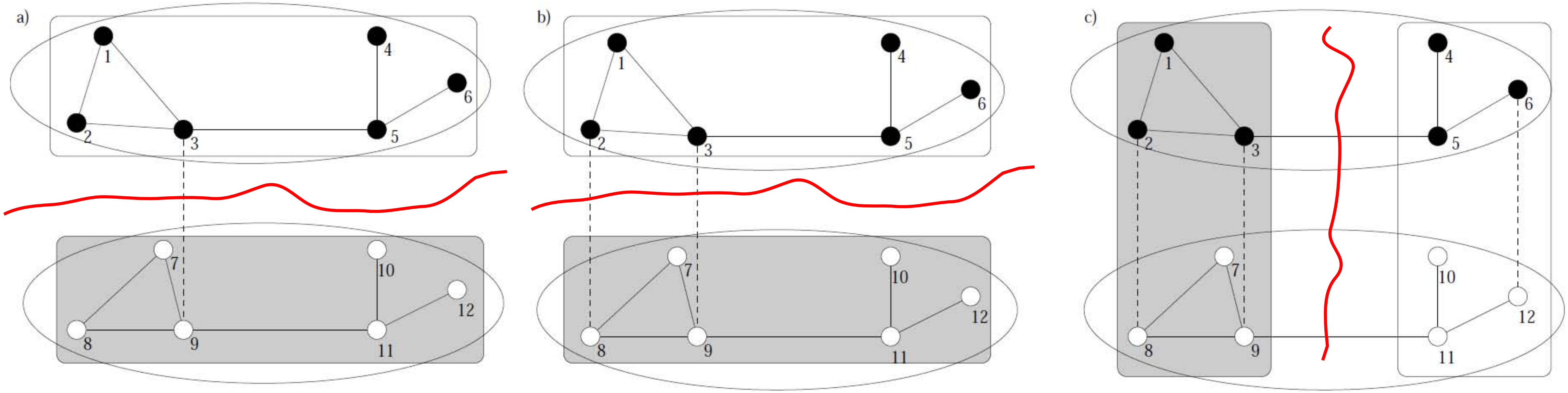
$$u_1^2=\Xi_1=(I,-I)$$ *The* ***inter-mode*** *cuts only interlinks* $\Lambda_2=2\alpha$

$$u_2^1=I\xi_2=(\xi_2,\xi_2)$$ *The* ***intra-mode*** *macro-eigenvector* λ_2



Inter/Intra Mode Transition HLS

Ellipses surround C-nets while rounded boxes indicate Fiedler partitions.
The read line indicates the (Fiedler cut.



These cuts remove only interlinks → "Intermodes of size 1 and 2

This cut removes only intralinks → "Intramode of size 2

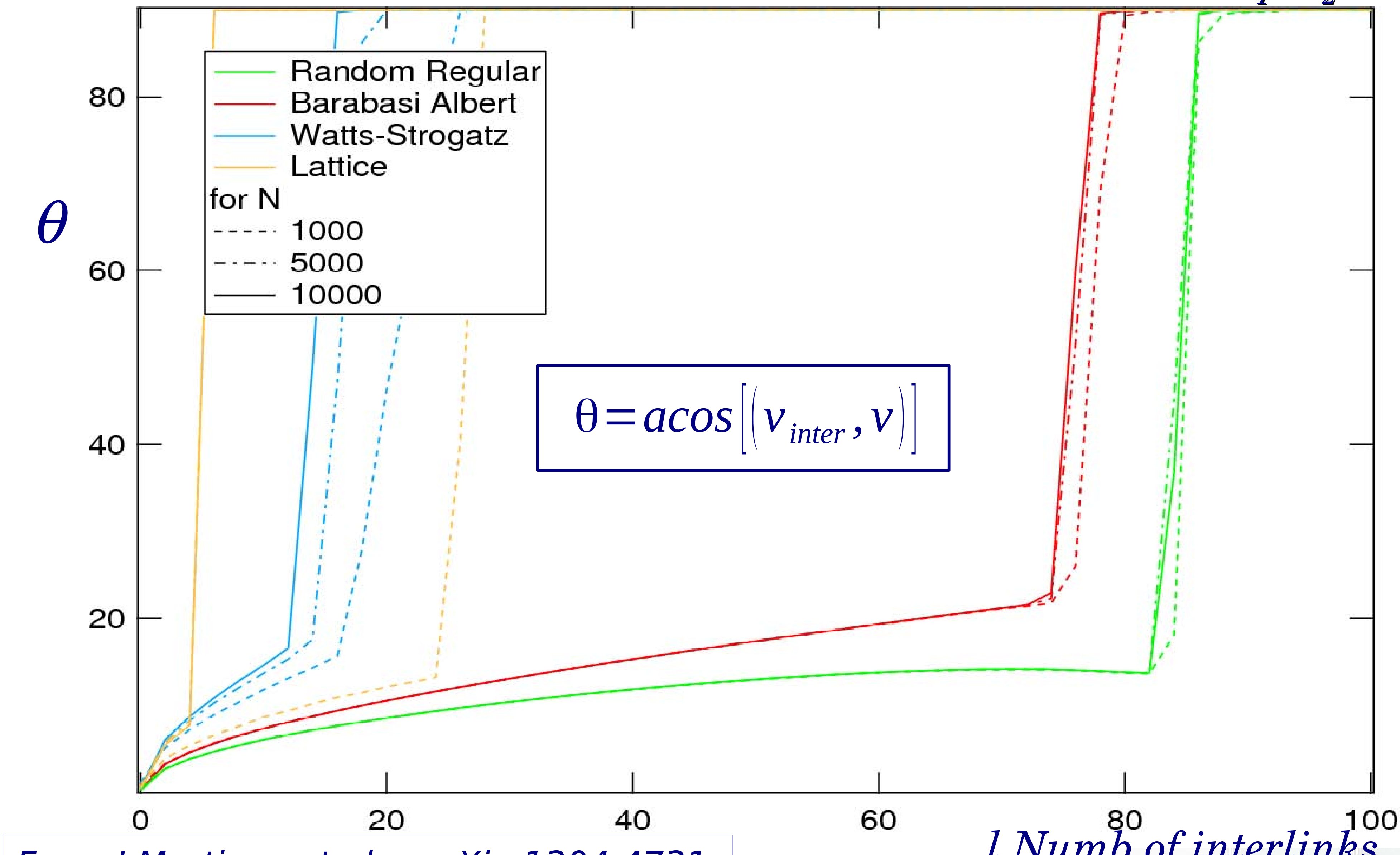
There is a critical amount of links beyond which the mode turns from inter to intra
At this value the two eigenvalue coincide:

$$\mu_1^2 = \mu_2^1 \qquad \Lambda_2 \alpha_c = \lambda_2$$

$$\alpha_c = \frac{\lambda_2}{\Lambda_2} \qquad l_c = N \frac{\lambda_2}{\Lambda_2}$$

HLS: Interdependence Angle

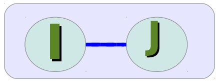
M=2 numerical simulations – 100 confs each – ***Identical Cnets $L_1=L_2$***



From J Martinez et al. arXiv:1304.4731

Mean-Field for RLS

The Macro-topology is trivial



$$A=\begin{pmatrix} 0\rightarrow A_1 & 1\rightarrow B_{12} \\ 1\rightarrow B_{21} & 0\rightarrow A_2 \end{pmatrix} \qquad L=\begin{pmatrix} L_1+\delta_1 & -B_{12} \\ -B_{21} & L_2+\delta_2 \end{pmatrix}$$

$$J=\begin{pmatrix} 1 & 1 & \dots & 1 \\ 1 & 1 & \dots & 1 \\ \dots & \dots & \dots & \dots \\ 1 & 1 & \dots & 1 \end{pmatrix} \qquad L=\begin{pmatrix} L_1+\beta N & -\beta J \\ -\beta J & L_2+\beta N \end{pmatrix}$$

RLS Mean Field

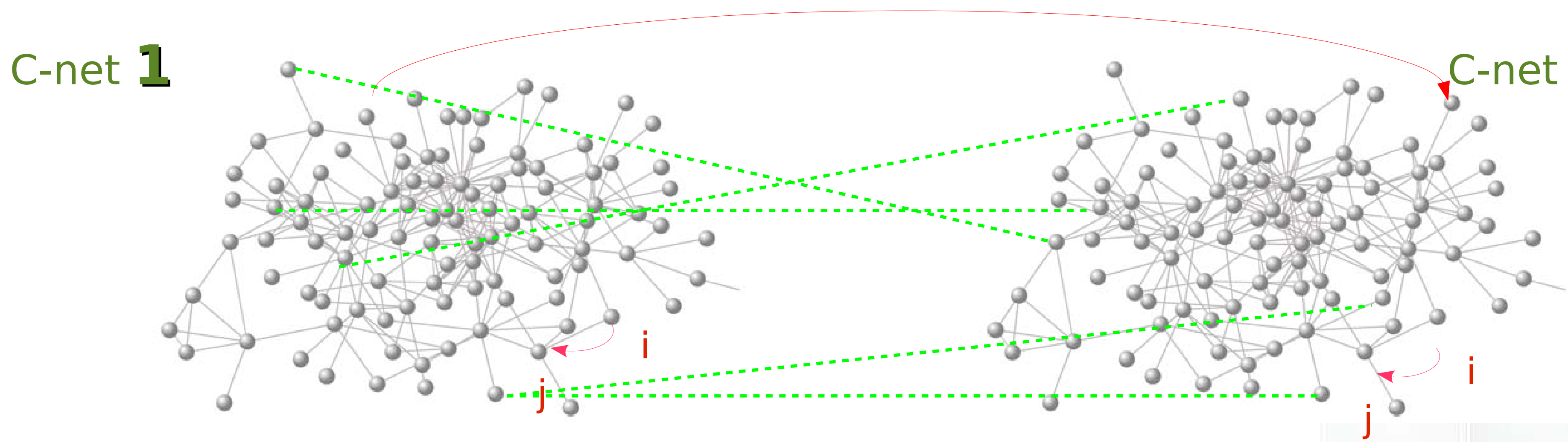
$$u_1^2=\Xi_1=(I,-I)$$

The ***inter-mode*** *macro-eigenvector* $\Lambda_2=2$

$$u_2^1=(\xi_2,\xi_2);(\xi_2,0);(0,\xi_2)$$

The ***intra-mode*** *macro-eigenvector* $\Lambda_1=1$

*The Random Linkage Strategy R****LS***



Mean-Field Approach RLS

$$E[\mu(L)] \sim \mu(E[L])$$

$$J = \begin{pmatrix} 1 & 1 & \dots & 1 \\ 1 & 1 & \dots & 1 \\ \dots & \dots & \dots & \dots \\ 1 & 1 & \dots & 1 \end{pmatrix} \qquad L = \begin{pmatrix} L_1 + \beta N & -\beta J \\ -\beta J & L_1 + \beta N \end{pmatrix}$$

RLS

Again L_0 commutes with L_I and hence a common set of eigenvectors exists:

$$v_i^I = \Xi^I \xi_i \qquad L_1 \xi_i = \lambda_i \quad L_{macro} \Xi_I = \Lambda_I \quad \lambda_1 = 0, \lambda_2 = \mu(C - net_1), \dots$$

$$\mu_i^I = \Lambda^I + \lambda_i \qquad \mu_1^1 = 0, \mu_1^2 = \Lambda_2 \beta, \mu_2^1 = \lambda_2 + \beta, \dots$$

K_{min} is minimum of macro-degrees i.e. minimum of interacting networks

$v^{inter} = v_1^2 = \Xi^2 I$ *This cuts only interlinks → "Intermode"*

$v_{intra} = \delta^{IJ} \xi_2$ *These cut only intralinks → "Intramode"*

$$\beta_c = \frac{\lambda_2}{(\Lambda_2 - 1) N} \qquad l_c = N \frac{\lambda_2}{\Lambda_2 - 1}$$

$$\alpha_c^{RLS} = \frac{\lambda_2}{\Lambda_2 - 1} = \alpha_c^{HLS} \frac{\Lambda_2}{\Lambda_2 - 1}$$

Critical Points

M=2 numerical simulations – 100 confs each – ***Identical C-nets L1=L2***

The ***slope change*** *corresponds to the "Intra-Iter" phase change*

	α Critical Values	
Model C-net	HLS	RLS
RR	0.85	2.10
BA	0.75	1.60
WS	0.25	0.40
2-D Lattice	0.30	0.50

$$\alpha_c^{RLS} \sim 2\,\alpha_c^{HLS}$$

In good agreement with he mean-field Prediction or M=2

$$\alpha_c^{RLS} = \alpha_c^{HLS} \frac{\Lambda_2}{\Lambda_2 - 1} = 2\,\alpha^{HLS}$$

From J Martinez et al. arXiv:1304.4731

Some general Results for Inter-modes

*An **Inter**-mode is Constant over C-nets:*

$$u^{(0)}_{inter} = \frac{1}{\sqrt{(N)}}(C_1, C_{1,}, .., C_1; C_2, C_{2,}, .., C_2, C_3, ..., C_M, C_M, ..., C_M) \qquad \sum_{I=1,M} C_I = 0 \qquad \eta = \frac{1}{\sqrt{(N)}}(1,1,..., 1)$$

$$\mu_{inter} = \min_{u_{inter}} \left| u^{(0)}_{inter}, L_I u^{(0)}_{inter} \right| \qquad \mu \le \mu_{inter} \qquad \mu_{inter} = \Lambda_2 (L^{eff})$$

$$A^{eff}_{IJ} \rightarrow A^{macro}_{IJ} \frac{l_{IJ}}{N} \qquad \sum_{ij=1,N} (\eta, B^{IJ}_{ij} \eta) = \frac{l_{IJ}}{N} \qquad L^{eff}_{IJ} = \delta_{IJ} \sum_{K=1,M} A^{macro}_{IK} - A^{eff}_{IJ}$$

*In the **thermodynamic limit** $N \rightarrow \infty$, L_{IJ} tend with **probability one** to its expectation value $\alpha_{IJ} = E[l_{IJ}/N]$.*

$$\frac{l_{IJ}}{N} \rightarrow \alpha_{IJ} \qquad A^{eff}_{IJ} \rightarrow A^{macro}_{IJ} \alpha_{IJ}$$

*When the linkage strategy is **uniform**:* $\alpha_{IJ} = \alpha$

$$\mu_{inter} = \Lambda_2 (L^{eff}) = \alpha \Lambda^{macro}_2$$

Some general Results for Intra-modes

*An **Intra**-mode cuts only intralinks:*

$$u^{(0)}_{intra} = \frac{1}{\sqrt{(M)}} (\xi_2 ; \xi_2 ; \ldots , \xi_2) \quad Ls = \langle L_I \rangle = \frac{1}{M} \sum_{I=1,M} L_I \qquad L_s \xi_{intra} = \lambda_2^{eff} \xi_{intra}$$

$$\mu \leq \left| u^{(0)}, (L_0 + L_I) u^{(0)} \right| = \lambda_2 + \sum_{I=1,M;J=1,M} \left[\sum_{i=1,N,j=1,N} B_{ij}^{IJ} (xi_2)_i^2 - (\xi_{2,} B^{IJ} \xi_2) \right]$$

$$\varphi^2 \stackrel{\text{def}}{=} \sum_{i=1,N,j=1,N} B_{ij}^{IJ} (\xi_2)_i^2 - (\xi_{2,} B^{IJ} \xi_2) \geq 0.$$

$$\mu_{intra} = \lambda_2^{eff} + \varphi^2 \leq \lambda_2^{eff}$$

Iff the Linkage Strategy is Homogeneous

$$\varphi^2 = \sum_{i=1,N,j=1,N} B_{ij}^{IJ} (\xi_2)_i^2 - (\xi_{2,} B^{IJ} \xi_2) \equiv 0 \qquad \mu_{intra} = \lambda_2^{eff}$$

For the random linkage strategy in the thermodynamic limit:

$$\mu_{intra} = \lambda_2^{eff} + \alpha$$

$$\left(B^{II} \xi_2 \rightarrow E[B^{II}] \xi_2 = J \xi_2 = 0. \right)$$

Phase transition general Results

Phase "inter"

$$\mu_{inter} \leq \mu_{intra}$$

$$\mu_{inter} = \Lambda_2 (L^{eff})$$

Phase "intra"

$$\mu_{inter} \geq \mu_{intra}$$

$$\mu_{intra} = \lambda_2^{eff} + \varphi^2 \leq \lambda_2^{eff}$$

Phase transition:

$$\mu_{inter} = \mu_{intra}$$

$$\Lambda_2^{intra} = \lambda_2^{eff} + \varphi^2$$

The Critical Point depends on the linkage strategy. Uniform examples:

Homogeneous

$$\alpha_c^{HLS} = \frac{\lambda_2^{eff}}{\Lambda_2}$$

Random

$$\alpha_c^{RLS} = \frac{\lambda_2^{eff}}{(\Lambda_2 - 1)}$$

Perturbation Theory

$$L = L_0 + g L_I$$

$$(d_I)_{ij} = \delta_{ij} \sum_{J=1}^{M} A_{IJ}^{macro} \sum_{k=1}^{N} (B_{IJ})_{ik}$$

$$(L_I)_{ij} = \delta_{ij} \sum_{k=1}^{N} (A_I)_{ik} - (A_I)_{ij}$$

$$\begin{cases} \mu = \sum_{k=1,N} \mu^{(k)} g^k = \mu^{(0)} + g\mu^{(1)} + g^2\mu^{(2)} + \ldots \\ u = \sum_{k=1,N} u^{(k)} g^k = u^{(0)} + g u^{(1)} + g^2 u^{(2)} + \ldots \end{cases}$$

$$\begin{pmatrix} L_1 + g d_1 & -g B_{12} & \ldots & -\%g B_{1M} \\ -g B_{21} & L_2 + g d_2 & \ldots & -g B_{2M} \\ \ldots & \ldots & \ldots & \ldots \\ -g B_{M1} & -g B_{M2} & \ldots & L_M + g d_M \end{pmatrix} = \begin{pmatrix} L_1 & 0 & \ldots & 0 \\ 0 & L_2 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & L_M \end{pmatrix} + g \begin{pmatrix} d_1 & -B_{12} & \ldots & -B_{1M} \\ -B_{21} & d_2 & \ldots & -B_{2M} \\ \ldots & \ldots & \ldots & \ldots \\ -B_{M1} & -B_{M2} & \ldots & d_M \end{pmatrix}$$

Characteristic Equations
(min-max theorem)

$$\begin{cases} L u = \mu u \\ (u, u) = 1 \\ (\eta, u) = 0 \end{cases}$$

$$\eta = \frac{1}{\sqrt{(N)}} (1,1,\ldots,1)$$

Perturbative Equations

$$\begin{cases} L_0 u^{(k)} + L_I u^{(k-1)} = \sum_{i=0,k} \mu^{(i)} u^{(k-i)} & \forall k \\ \sum_{i=0,k} (u^{(k-i)}, u^i) = 0 & \text{for } k \geq 1 \quad (u^{(0)}, u^{(0)}) = 1 \\ \sum_{i=0}^{k} (u^{(k-i)}, \eta) = 0 & \forall k \end{cases}$$

Perturbation Theory Basic results

$$\mu = \sum_{k=1,N} \mu^{(k)} g^k = \mu^{(0)} + g\mu^{(1)} + g^2\mu^{(2)} + \ldots$$

For small g, the inter-mode dominates (Hp uniform linkage):

$$\mu = \mu_0 + g\mu^{(1)} + g^2\mu^{(2)} + \ldots = g\alpha\Lambda_2 + g^2(u^{(1)}, L_I u_{inter}^{(0)}) + \ldots$$

$$u = u^{(0)} + g u^{(1)} + \ldots = u_{inter} + g u_{inter}^{(1)} + \ldots \qquad L_0 u^{(1)} = (\mu^{(1)} - L_I) u_{inter}^{(0)}$$

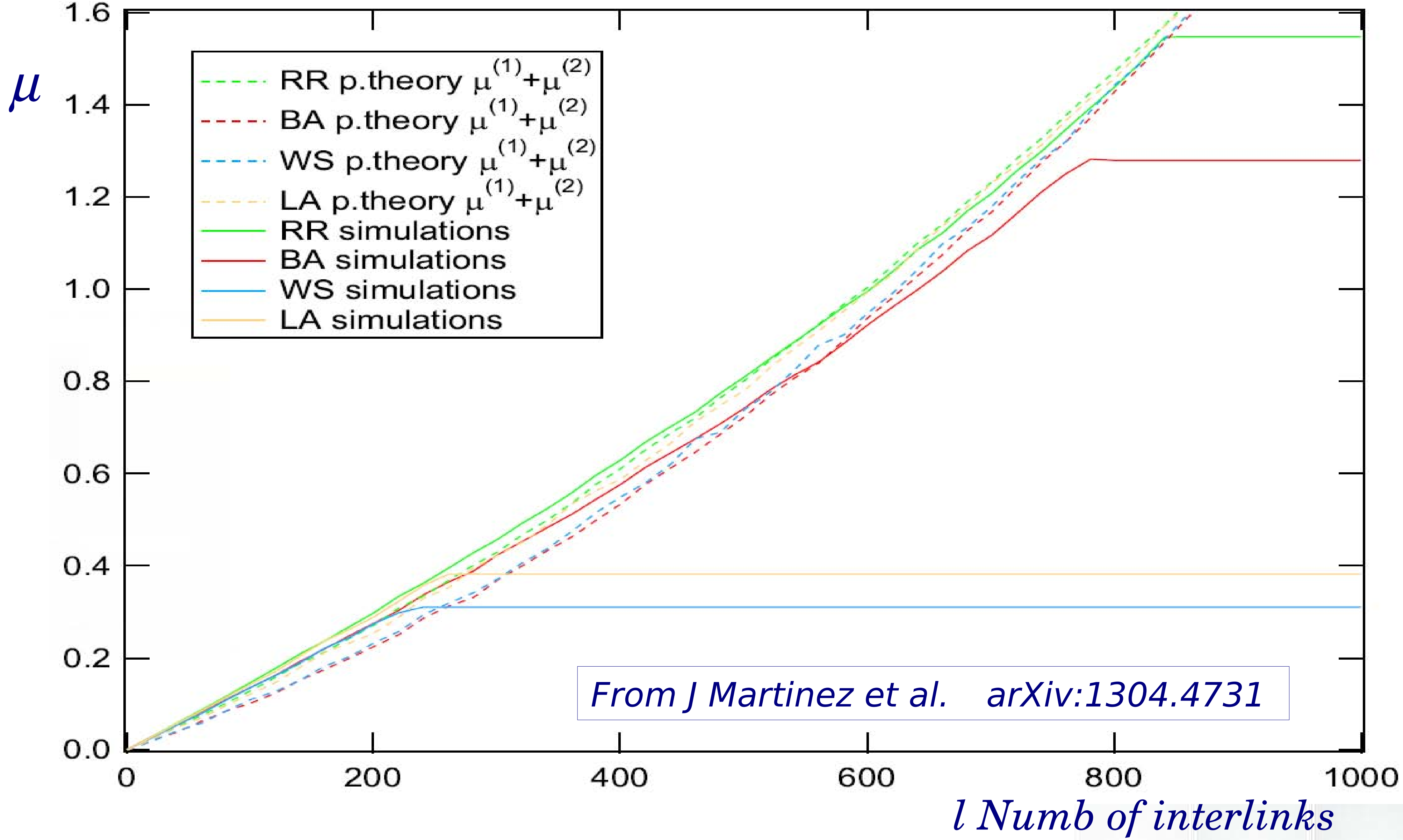
For larger g, the other modes may compete (Hp uniform linkage):

$$\mu = \mu^{(0)} + g\mu^{(1)} + g^2\mu^{(2)} + \ldots = \lambda_2 + g(\alpha + \varphi^2) + g^2(u^{(1)}, L_I u^{(0)}) + \ldots$$

$$u = u^{(0)} + g u^{(1)} + \ldots = u_{intra} + g u_{intra}^{(1)} + \ldots$$

Homologous Linkage Perturbation

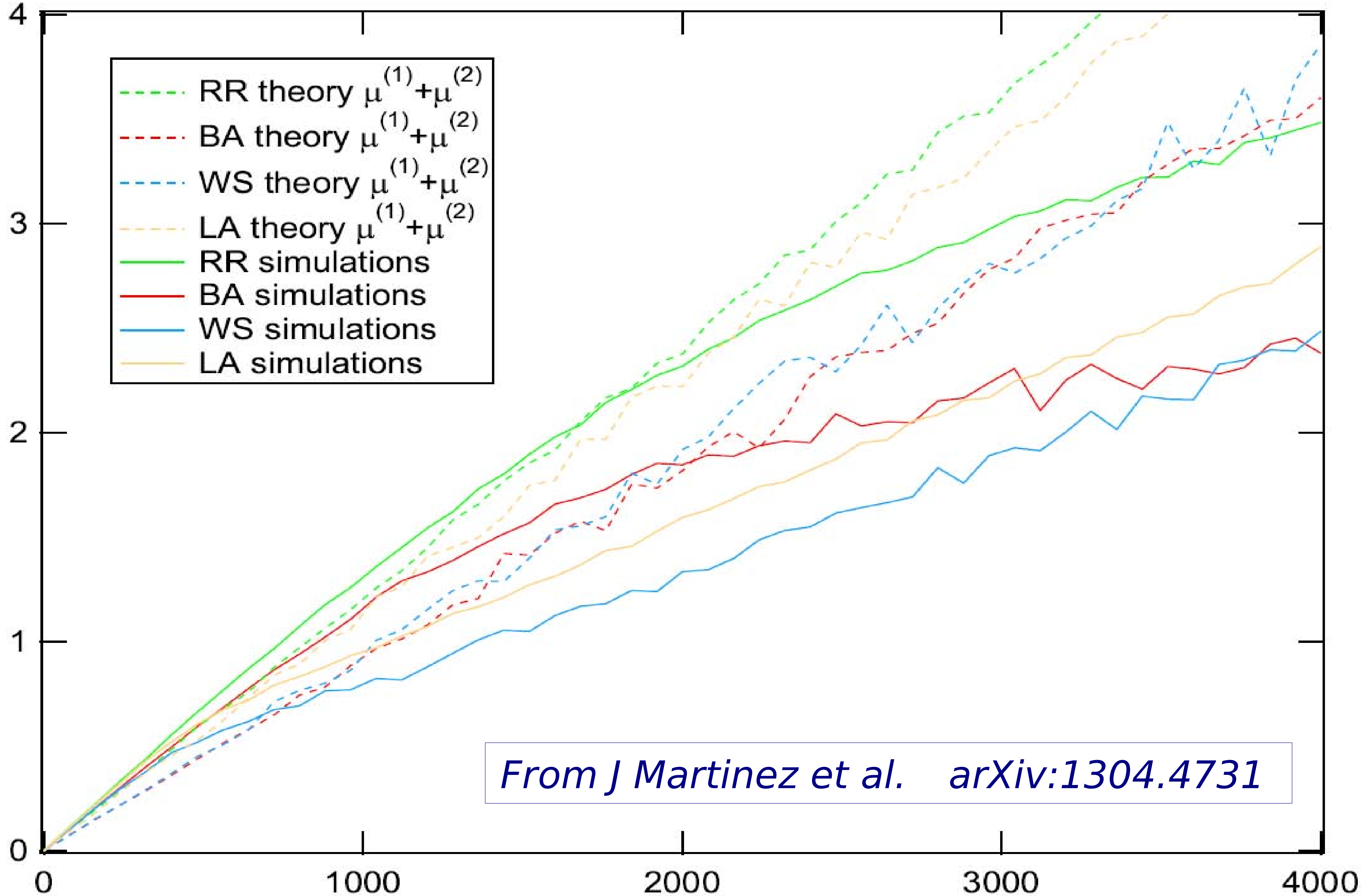
M=2 numerical simulations N=1000 – 100 confs each



Random Linkage Perturbation

M=2 numerical simulations N=1000 – 100 confs each

μ



l Numb of interlinks

Conclusions

The ***diffusion*** *processes, the* ***algebraic cut*** *and the* ***synchronization*** *of a Network of Networks do exhibit two different phases: "I****ntra****" and "I****nter****". Depending on the number or interlinks the System of Systems swaps from one phase to an other.*

A set of results have been presented to provide a theoretical frame to the general phenomenon. Exactly solvable models, Mean-Field approximation and Perturbation Approaches provide different means to predict the transition and the ***critical density of interlinks****.*

The ***Topologies*** *of both the macroscopic Network and the component Networks, together with the "****Linkage Strategy****" are solely responsible for the phase transition and the value of the critical density of interlinks.*

Special Linkage Strategies (Homologous and General Linkage) have been directly inspected by numerical calculation.

”The Synchronization of Interdependent Networks Undercomes a Phase Transition”

“Thanks for your attention”

You are invited

to

NetoNets2013

Copenhagen - June 3/4-th

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