Egher an Turki Shirazi
Bedest ared dili mara,
Be khali hinduish bakhshem
Samarcand u **Bokharara**.

Would that Shirazi Turk behold our heart; then,
I'll gift, to her Indian Mole, both **Samarkand** and **Bokhara**.
Pour the remaining **wine**, Saghi! — in paradise you shall not find
the river banks so firm — nor the pleasure of a prayer rug.
These impudent **beauties of the city of confusion** steal
patience from my heart, like a **Khan in a joyous plunder**.
The face of the beloved is pleased with our unconsummated love:
what need the beauteous face has of earth, water, or art?
From the ever.increasing **beauty of Joseph**, this I understood:
love rends the curtain of virtue from **Zoleykha**'s face.
If you curse — if you abuse me, I will pray for you:
bitter response suits the ruby lips of the sweetest heart.
My love: more precious than life the lucky youth
holds the advice of the virtuous sage.
Come, sing of **wine** and **minstrels** — seek less the secrets of life;
none has solved — nor can — this enigma with the logical mind.
Hafez, you sang **ghazal**, made pearls of words; come and sing:
the Universe graces your verse with a marriage to the Pleiades.

Quantum Graph & Quantum Filter

Taksu Cheon
Kochi University of Technology, Japan

NATO Conference@Samarkand, 23.5.2013

QUANTUM GRAPH

- Quantum particle on graph (lines & nodes)

  Mathematical physics aspect

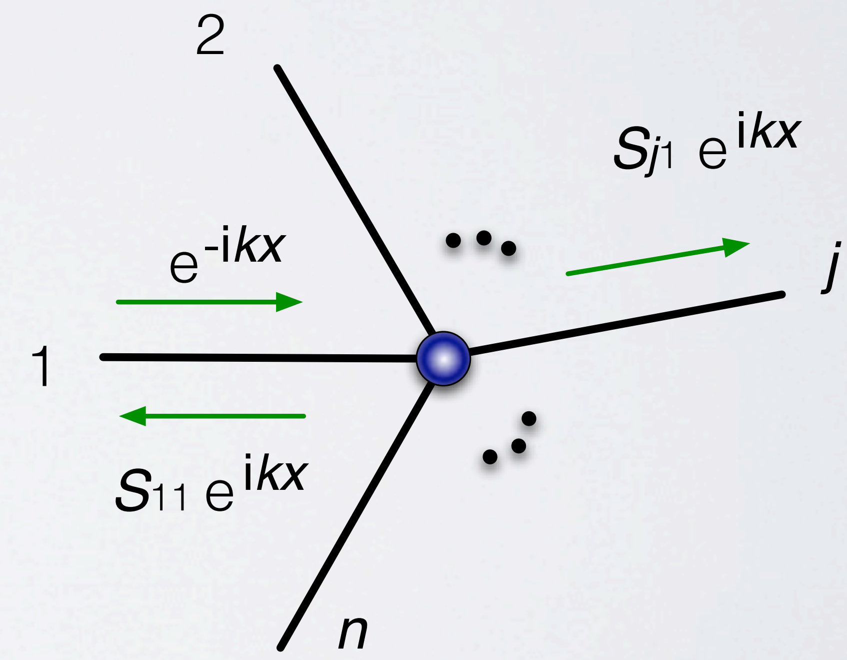
  - Minimal extension of 1-dim quantum mechanics

  "applicational" aspect

  - Model of single electron device

- Nontrivial physics
  with Exotic vertex

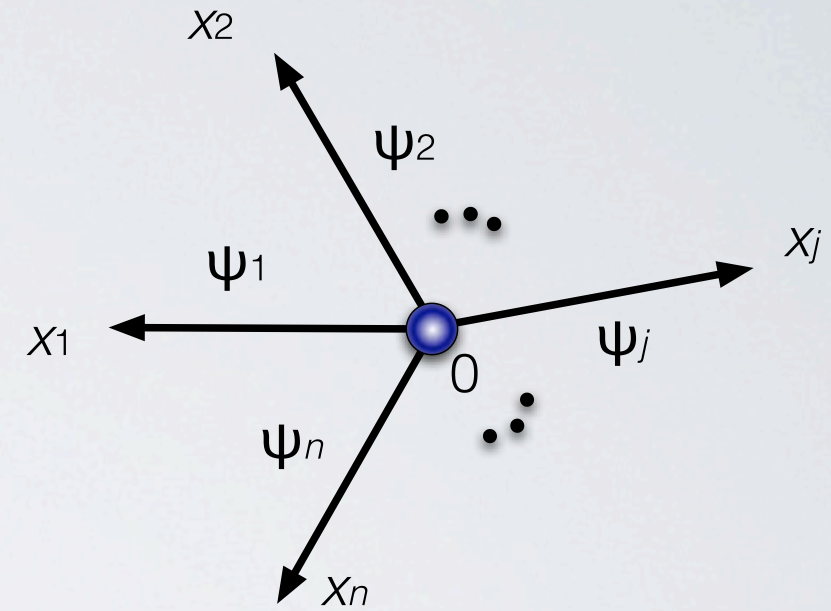
- Application to "quantum device"



CONNECTION CONDITION

- At the node, define

$$\Psi = \begin{pmatrix} \psi_1(0_+) \\ \vdots \\ \psi_n(0_+) \end{pmatrix} \quad \Psi' = \begin{pmatrix} \psi_1'(0_+) \\ \vdots \\ \psi_n'(0_+) \end{pmatrix}$$



- flux conservation $\psi^* \psi' = \psi'^* \psi$ (Self-Adjoint Extension)

$$A\,\psi + B\,\psi' = 0$$
$$\mathrm{rank}(A, B) = n, \ AB^* = BA^*$$

(Kostrykin & Schader, 99)

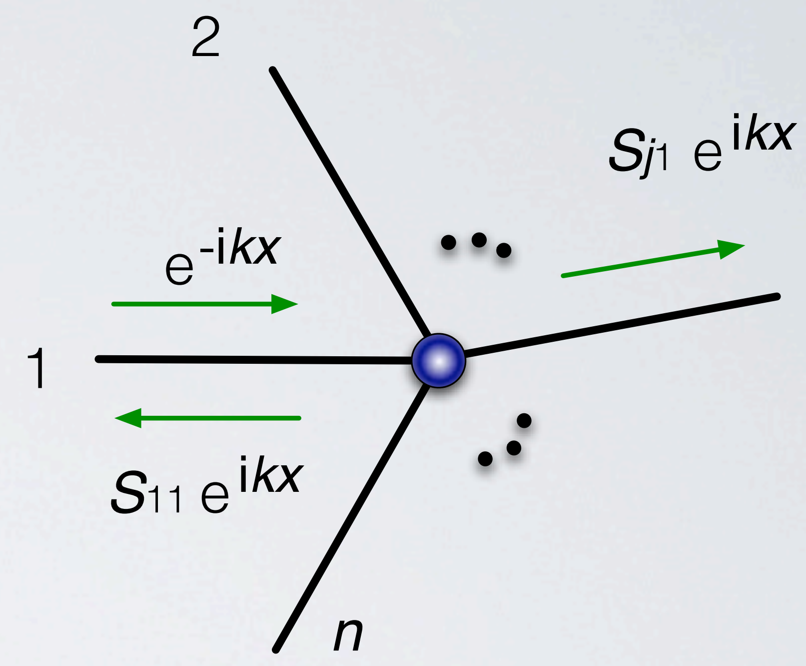
: n^2 parameters

- ψ, ψ' different dimension : scale anomaly

SCATTERING MATRIX

- Scattering from ***j***-th to ***i***-th lines

$$\psi_i^{(j)}(x_i) = e^{-\mathrm{i}kx_i} + \mathcal{S}_{ii}e^{\mathrm{i}kx_i} \qquad (i=j)$$
$$= \mathcal{S}_{ij}\, e^{\mathrm{i}kx_i} \qquad (i \neq j)$$



- Scattering matrix $S(k)$ from

$$A\,\psi + B\,\psi' = 0$$

$$(\Psi^{(1)} \cdots \Psi^{(n)}) = \mathcal{S}(k) + I$$
$$(\Psi'^{(1)} \cdots \Psi'^{(n)}) = ik\mathcal{S}(k) + ikI$$

—» $A(S(k) + I) + ik\, B(S(k) - I) = 0$

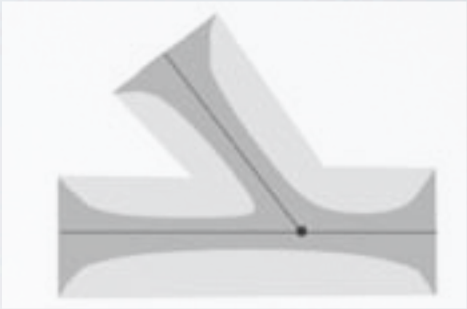
- $$\mathcal{S}(k) = -\frac{1}{A + \mathrm{i}kB}(A - \mathrm{i}kB)$$

FREE CONNECTION

- "normal" or "free" connection : thin tube limit

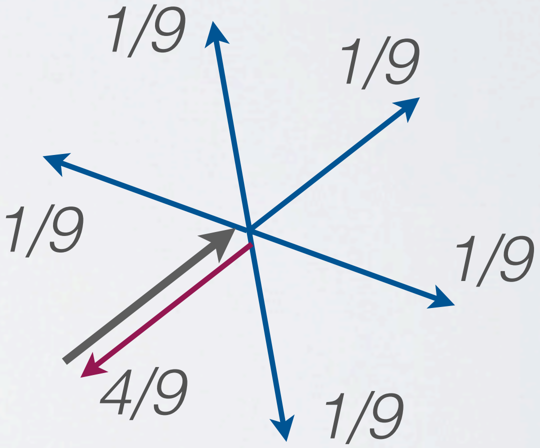
  $\psi_1(0) = \psi_2(0) = \ldots = \psi_n(0)$

  $\psi_1{}'(0) + \psi_2{}'(0) + \ldots + \psi_n{}'(0) = 0$



  transm. $S_{ij}(k) = -2/n$, refl. $S_{jj}(k) = 1-2/n$

- dual "free′" exists



  $\psi_1(0) + \psi_2(0) + \ldots + \psi_n(0) = 0$

  $\psi_1{}'(0) = \psi_2{}'(0) = \ldots = \psi_n{}'(0)$

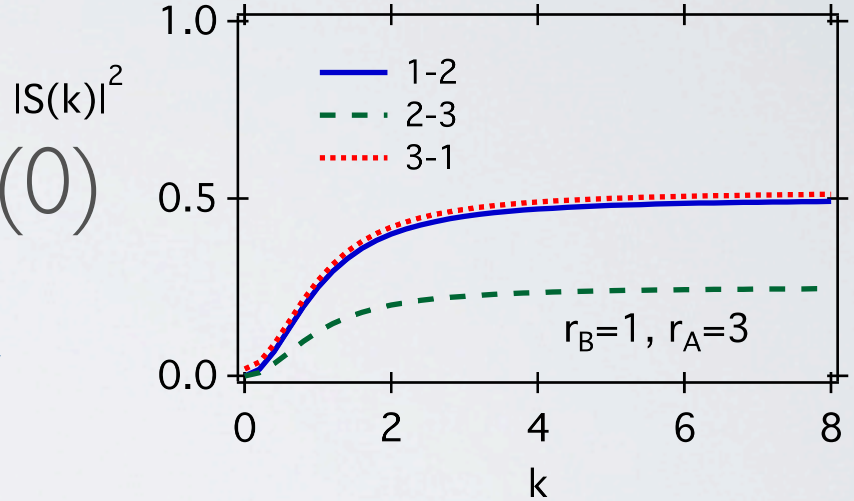
  transm. $S_{ij}(k) = 2/n$, refl. $S_{jj}(k) = -1+2/n$

total reflection at n \to ∞

DELTA & DELTA-PRIME VERTICES

- delta connection   free + obstacle v : [1/L] length scale

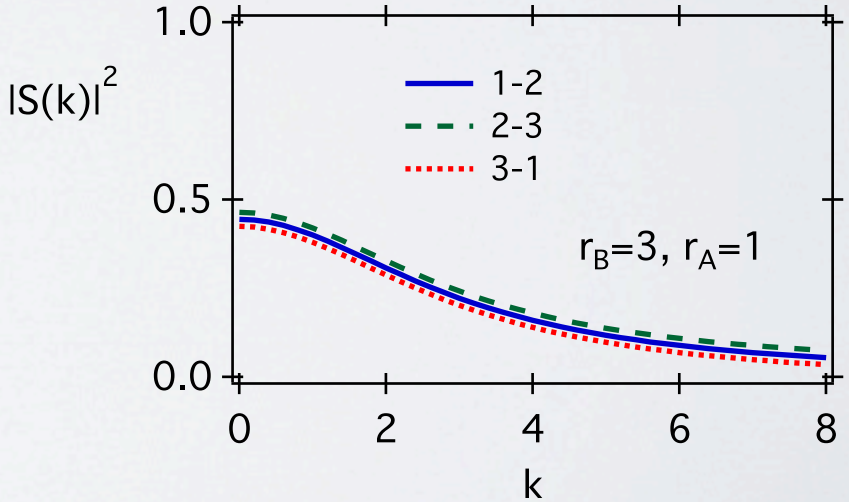
$$\psi_1(0) = \psi_2(0) = \ldots = \psi_n(0)$$
$$\psi_1'(0) + \psi_2'(0) + \ldots + \psi_n'(0) = v\,\psi(0)$$



discontinuous ψ' high-pass filter

- delta-prime connection u : [L]

$$\psi_1(0) + \psi_2(0) \ldots + \psi_n(0) = u\,\psi'(0)$$
$$\psi_1'(0) = \psi_2'(0) = \ldots = \psi_n'(0)$$



discontinuous ψ low-pass filter

SCALE INVARIANT VERTICES

- Extension of free node with scale invariant parameters
  $\psi_1(0) = 1/t_2\ \psi_2(0) = \cdots = 1/t_n\ \psi_n(0)$
  $\psi_1'(0) + t_2\ \psi_2'(0) + \cdots + t_n\ \psi_n'(0) = 0$

Branching ratio controlled by $|t_n|^2$

—» *k*-independent scattering (Fulop & Tsutsui '00)

- Their "dual" partners
  $\psi_1(0)' + t_2\ \psi_2(0) + \ldots + t_n\ \psi_n(0) = 0$
  $\psi_1'(0) = 1/t_2\ \psi_2'(0) = \ldots = 1/t_n\ \psi_n'(0)$

- Mixed types ($\{1,\ldots,m\}\ \{m+1,\ldots,n\}$) also exist ($m=1,\ldots,n$)

GENERAL VERTEX

- General KS connection cond. rewritten with $\{m, m'\}$ as

$$\begin{pmatrix} I^{(m)} & T \\ 0 & 0 \end{pmatrix} \Psi' = \begin{pmatrix} S & 0 \\ -T^\dagger & I^{(n-m)} \end{pmatrix} \Psi$$

or

$$\begin{pmatrix} \underline{S} & 0 \\ -\underline{T}^\dagger & I^{(n-m')} \end{pmatrix} \Psi' = \begin{pmatrix} I^{(m')} & \underline{T} \\ 0 & 0 \end{pmatrix} \Psi$$

S : delta [1/*L*]

S̲ : delta-prime [*L*]

T , ***T̲*** : FT param [1]

T : $m \times (n\text{-}m)$ complex, S : $m \times m$ Hermitian

$\underline{T}$: $m' \times (n\text{-}m')$ complex, $\underline{S}$: $m' \times m'$ Hermitian

$m + m' = n + s$; $s = \mathrm{rank}(S) = \mathrm{rank}(\underline{S})$

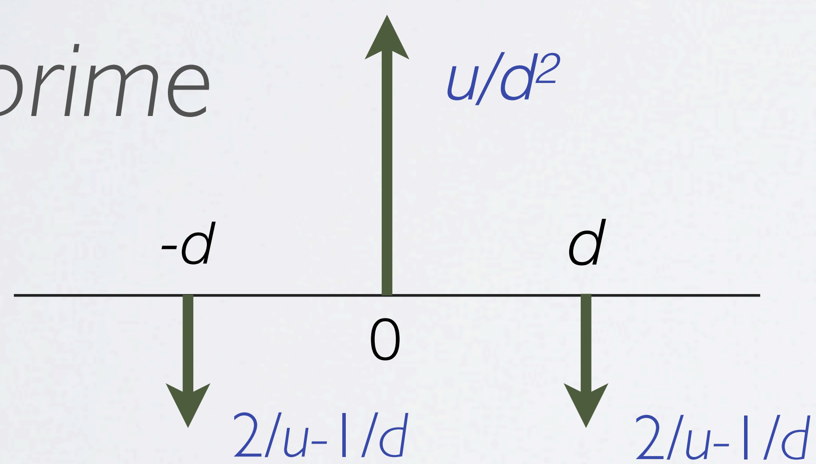
- —» finite construction of exotic vertex (Cheon, Exner, Turek, 2011)

DEMIENSIONAL TRANSMUTATION

- $n = 2$ *delta* $\qquad \psi_+' - \psi_-' = v\,\psi_+ = v\,\psi_- \qquad v : [1/L]$
- $n = 2$ *delta-prime* $\qquad \psi_+ - \psi_- = u\,\psi'_+ = u\,\psi'_- \qquad u : [L]$
- $n = 2$ *scale invariant* $\psi_+' = a\,\psi_-',\ \ \psi_+ = (1/a)\,\psi_- \qquad a : [1]$

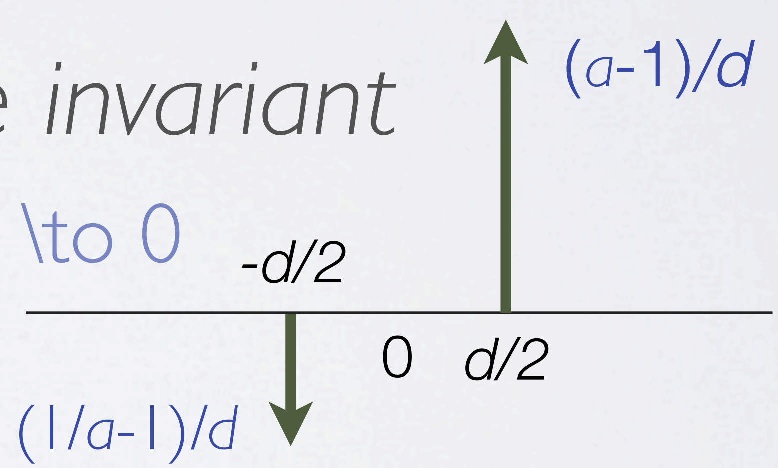
delta-prime

$d \to 0$



scale invariant

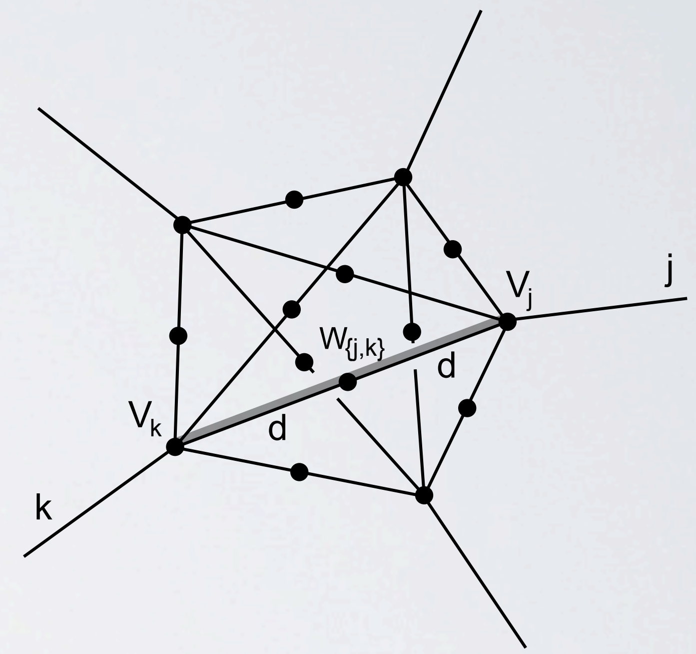
$d \to 0$



Emergence of scale with strength renormalization

EXOTIC VERTEX FROM DELTAS

- Finite approximation scheme for general vertex
  - -- cut node, connect all pairs by lines (j, k) of length d (\to 0)
  - -- n δs $[v_j]$ at new nodes,
  - -- $n(n\text{-}1)/2$ δs$[w_{jk}]$ at the center of (j, k)
  - -- $n(n\text{-}1)/2$ vector potentials $[A_{jk}]$ on (j, k)
  - -- $v_j = \gamma_j + \beta_j/d$, $A_{ij} = \eta/d$, $w_{jk} = \beta_{jk}/d + \eta_{jk}/d^2$

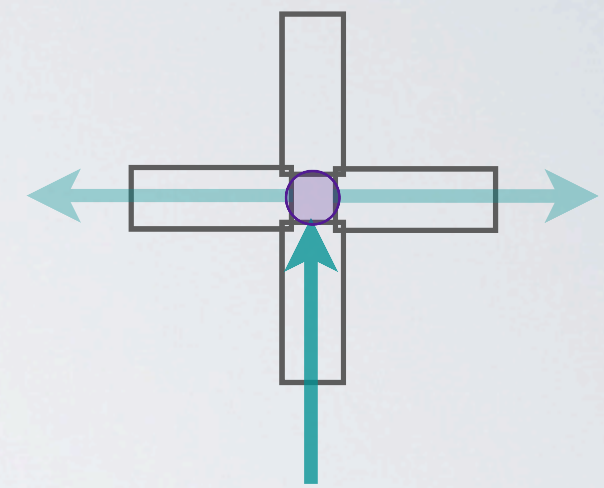


- Norm-resolvent convergence proved
  - $R^{Ap}(k^2)$: resolvent on $L^2(G_d)$; $G_d = (\mathbb{R}^+)^n \oplus (0,d)^{\frac{n(n-1)}{2}}$
  - $R(k^2)$: resolvent on $L^2((\boldsymbol{R}^+)^n)$
  - $R^E(k^2) = R(k^2) + 0$ $n + 2\, n(n\text{-}1)/2 = n^2$
  - $\| R^E(k^2) - R^{Ap}(k^2) \| \to 0_+$

COUNTER INTUITIVE COUPLING

- Scale invariant n=4 graph with $T = \begin{pmatrix} a & a \\ a & -a \end{pmatrix}$

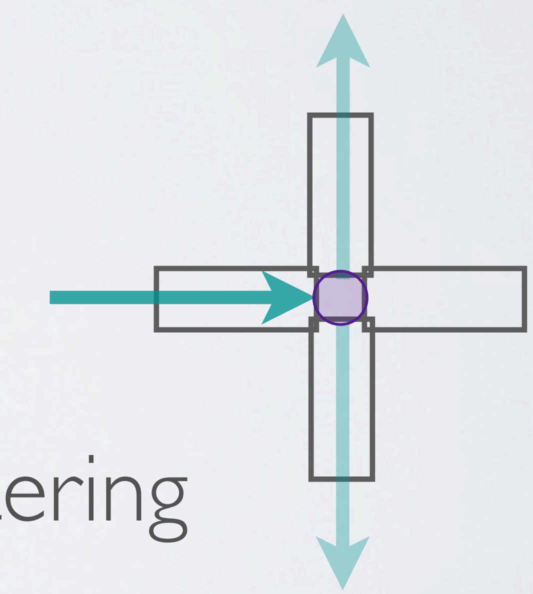
$$\mathcal{S} = \begin{pmatrix} \frac{1-2a^2}{1+2a^2} & 0 & \frac{2a}{1+2a^2} & \frac{2a}{1+2a^2} \\ 0 & \frac{1-2a^2}{1+2a^2} & \frac{2a}{1+2a^2} & -\frac{2a}{1+2a^2} \\ \frac{2a}{1+2a^2} & \frac{2a}{1+2a^2} & -\frac{1-2a^2}{1+2a^2} & 0 \\ \frac{2a}{1+2a^2} & -\frac{2a}{1+2a^2} & 0 & -\frac{1-2a^2}{1+2a^2} \end{pmatrix}$$



- In particular, with $a = \frac{1}{\sqrt{2}}$

$$\mathcal{S} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \\ 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{pmatrix}$$

pair-wise equiscattering
with no reflection



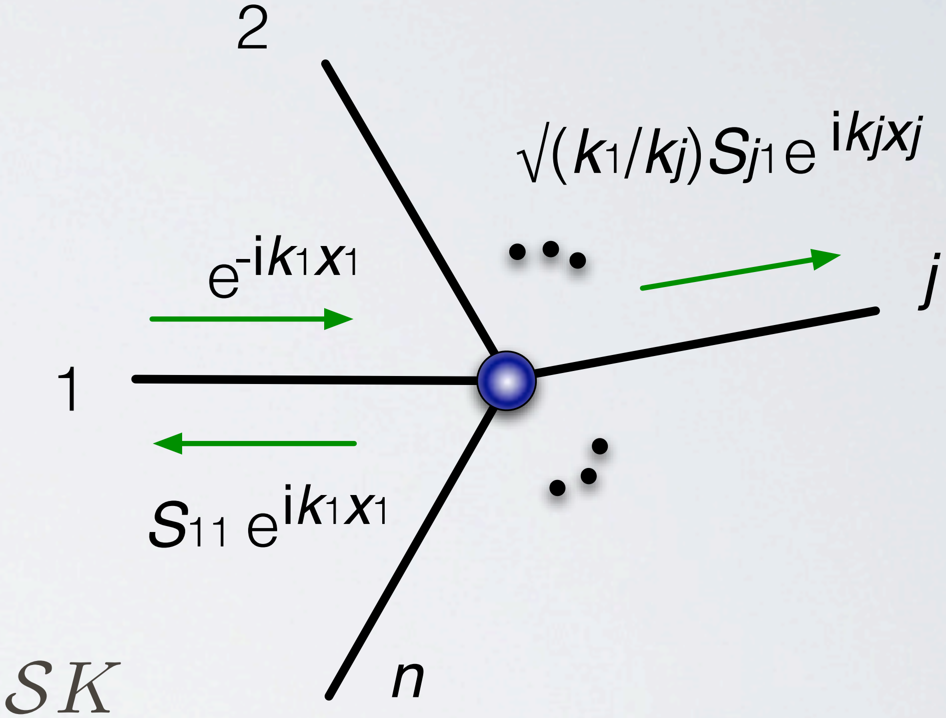
POTENTIALS ON LINES

- Potential U_j on line ***j***

$$k_j = \sqrt{k^2 - U_j}$$

$$\psi_i^{(j)}(x) = e^{-\mathrm{i}k_j x} + \mathcal{S}_{jj} e^{\mathrm{i}k_j x} \quad \text{for } i = j$$

$$= \quad \mathcal{S}_{ij} \sqrt{\frac{k_j}{k_i}} e^{\mathrm{i}k_i x} \quad \text{for } i \neq j$$



- Define $K = \{\sqrt{k_i}\delta_{ij}\}$

$$A\,\psi + B\,\psi' = 0$$

$$(\Psi^{(1)} \cdots \Psi^{(n)}) = I^{(n)} + K^{-1}\mathcal{S}K$$

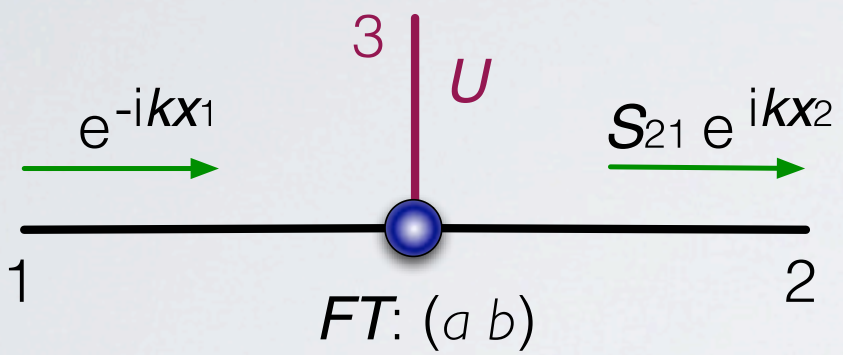
$$(\Psi'^{(1)} \cdots \Psi'^{(n)}) = \mathrm{i}K^2 + \mathrm{i}K\mathcal{S}K$$

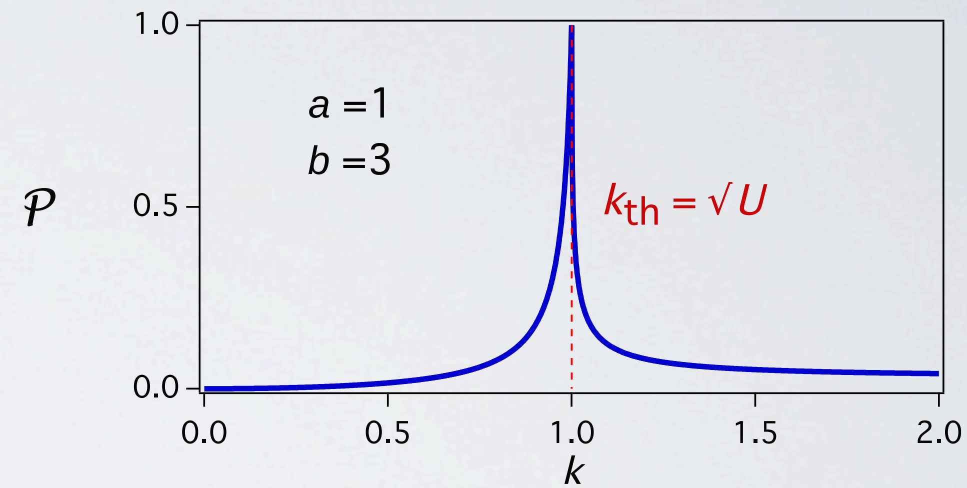
- Scattering matrix

$$\mathcal{S} = -(AK^{-1} + \mathrm{i}BK)^{-1}(AK^{-1} - \mathrm{i}BK)$$

THRESHOLD RESONANCE

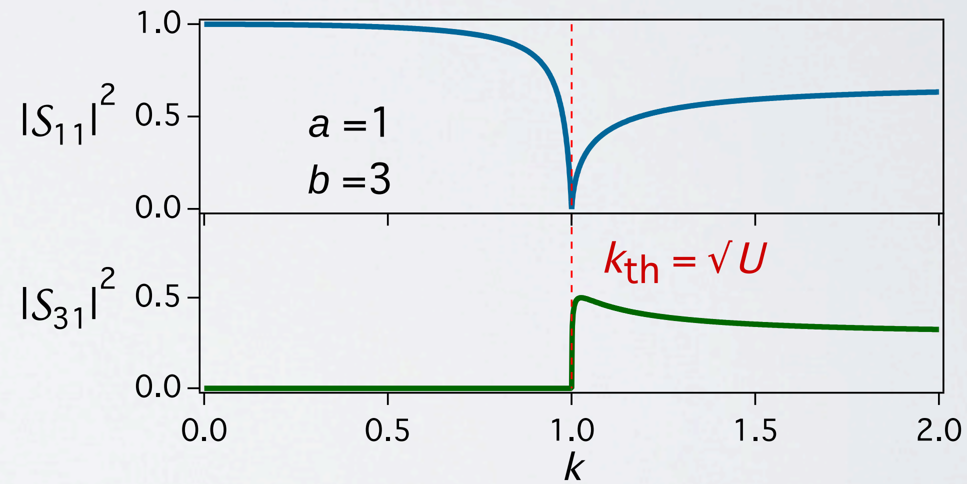
- N=3 Graph node $T = (a \quad b)$ with external field U





- 1–»2 transm. $P = |S_{21}|^2$

$$\mathcal{S}_{21}(k;U) = \frac{2a}{1 + a^2 + b^2\sqrt{1 - \frac{U}{k^2}}}$$



- $b >> a \geq 1$:
  threshold resonance at $k_{th} = \sqrt{U}$

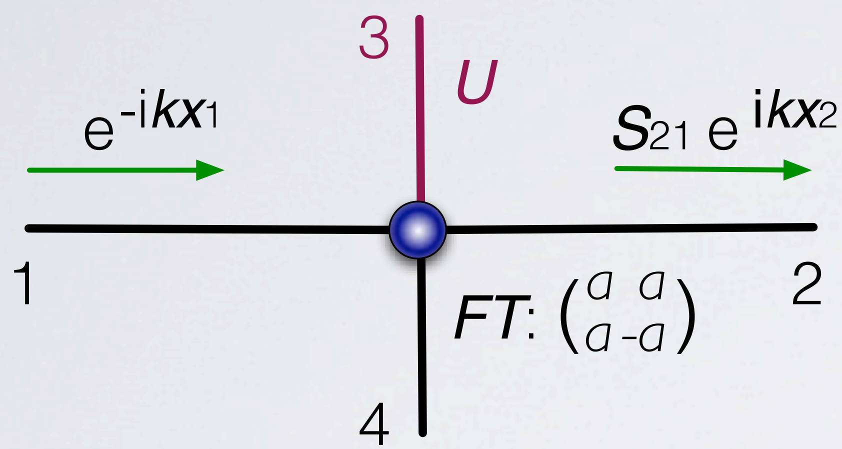
  U-controlled monochromatic filtering

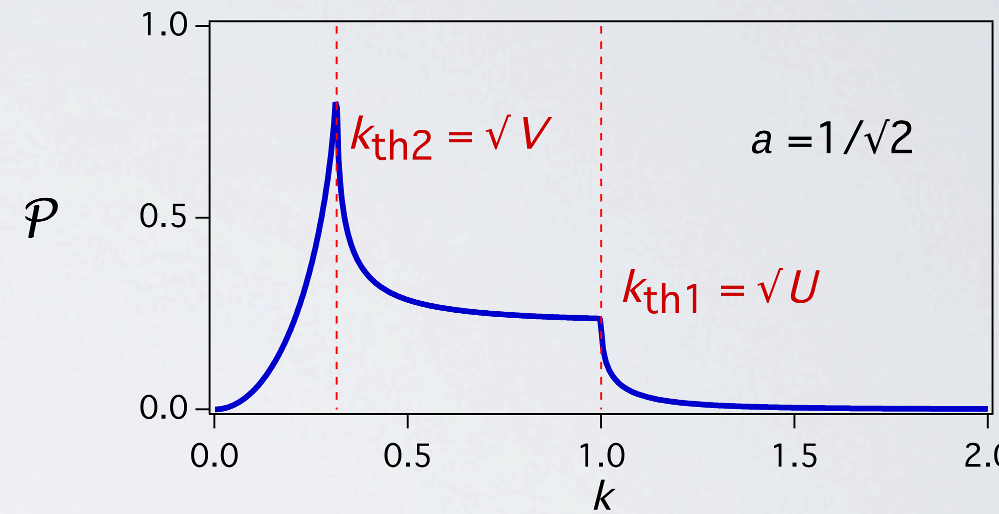
$$k_{\text{pol}} = \frac{b^2}{\sqrt{b^4 - (1+a^2)^2}}\sqrt{U}$$

pole on unphysical surface

CONTROLLABLE BAND FILTER

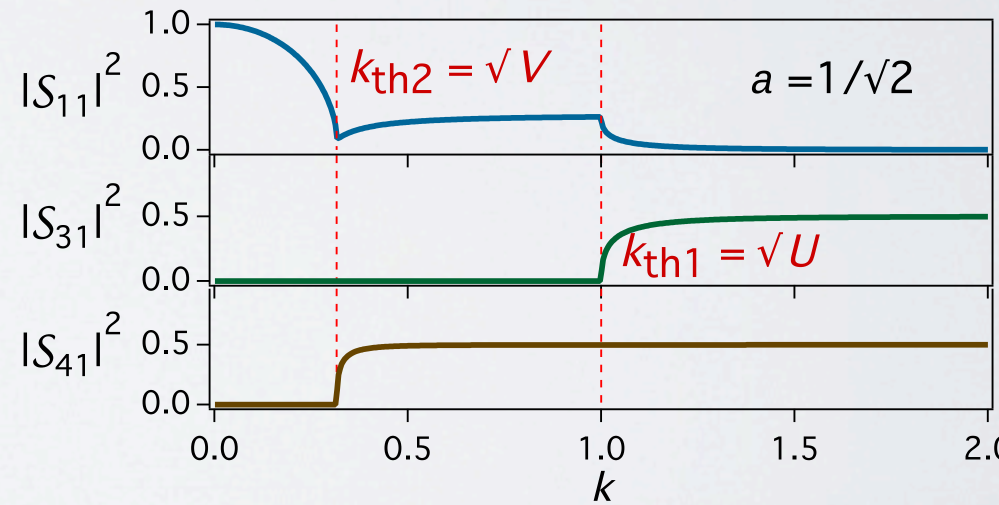
- N=4 Graph node $T = \begin{pmatrix} a & a \\ a & -a \end{pmatrix}$ with external field *U*, *V*





- 1—»2 transm. $P = |S_{21}|^2$

  Interference of 2 resonances



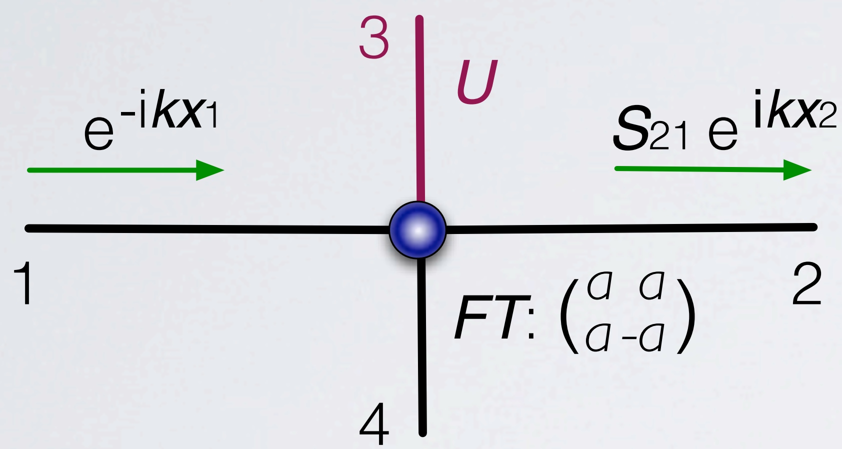
- No transm. when $V = U$

  $k^{(1)}{}_{th} = \sqrt{U}$, $k^{(2)}{}_{th} = \sqrt{V}$

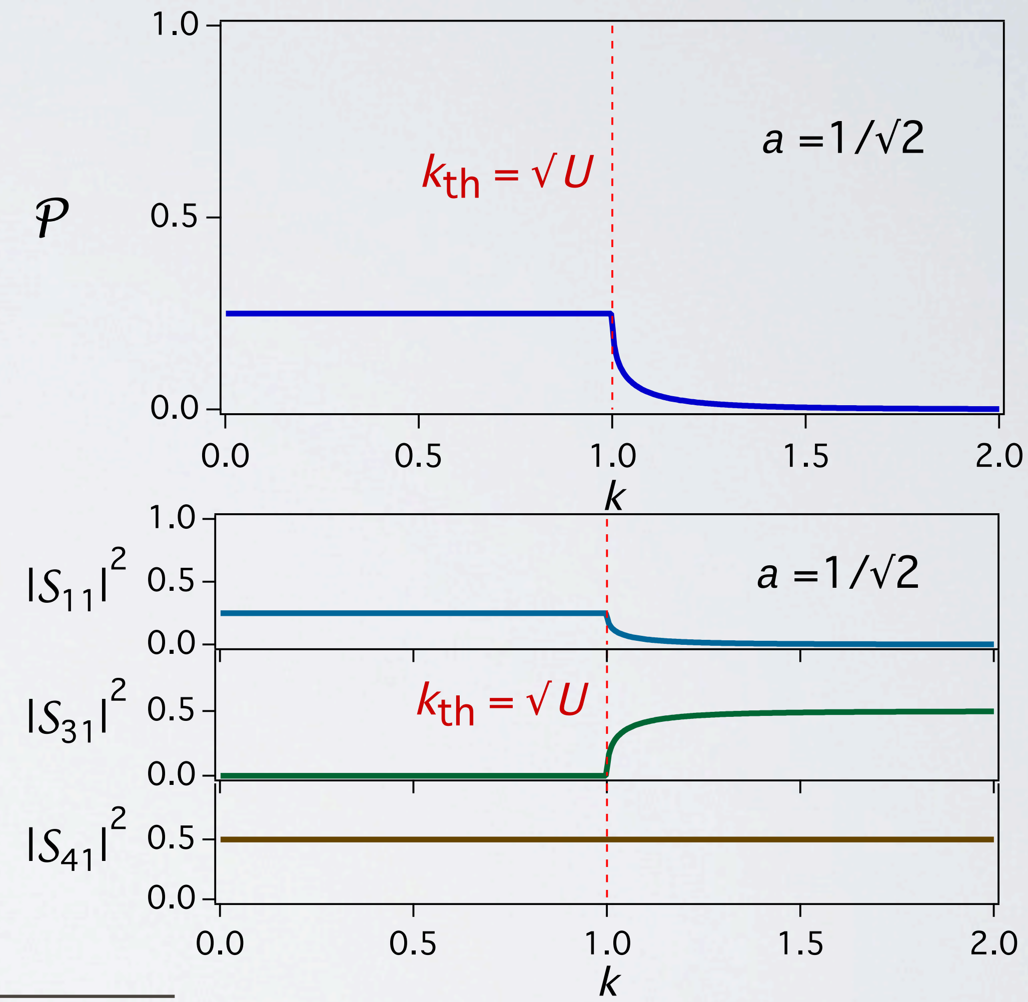
  —» [√U, √V] Band Filter

CONTROLLABLE FLAT FILTER

- Graph node $T = \begin{pmatrix} a & a \\ a & -a \end{pmatrix}$ with external field U



$$a = \frac{1}{\sqrt{2}}$$



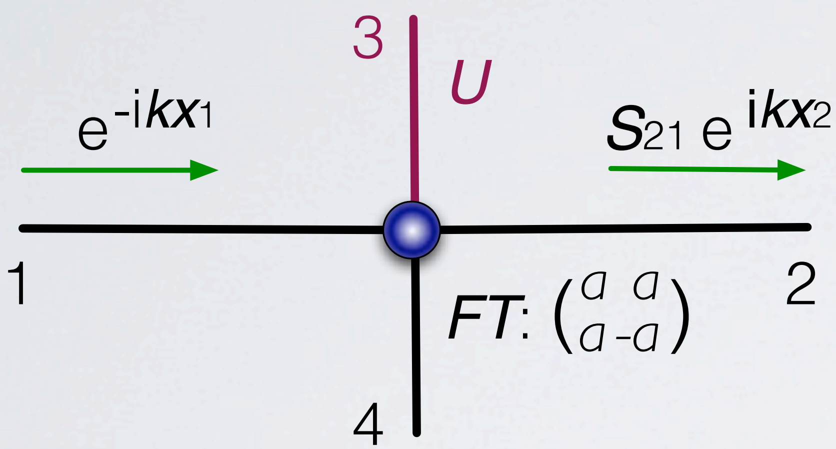
- 1–»2 transm. $P = |S_{21}|^2$

$$\mathcal{S}_{21}(k;U) = \frac{2a^2\left(1-\sqrt{1-\frac{U}{k^2}}\right)}{(1+2a^2)+2a^2(1+2a^2)\sqrt{1-\frac{U}{k^2}}}$$

- $[0,\sqrt{U}]$ flat band-filter

CONTROLLABLE FLAT FILTER

- Graph node $T = \begin{pmatrix} a & a \\ a & -a \end{pmatrix}$ with external field U



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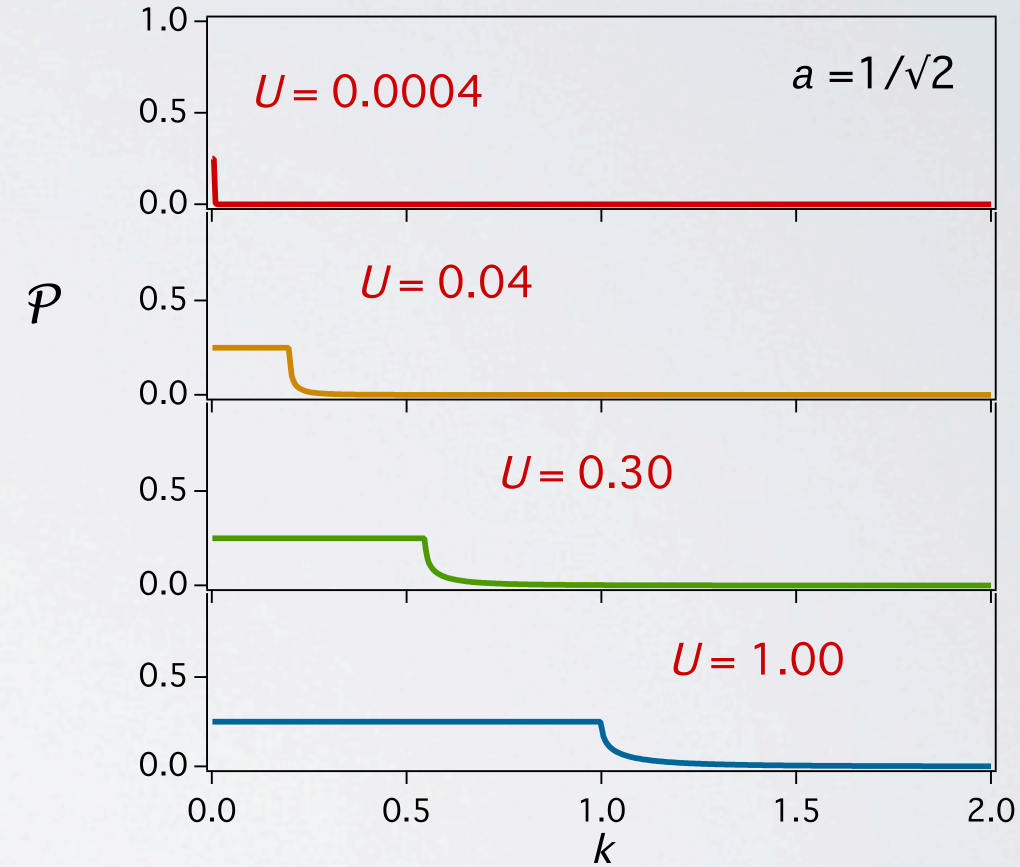

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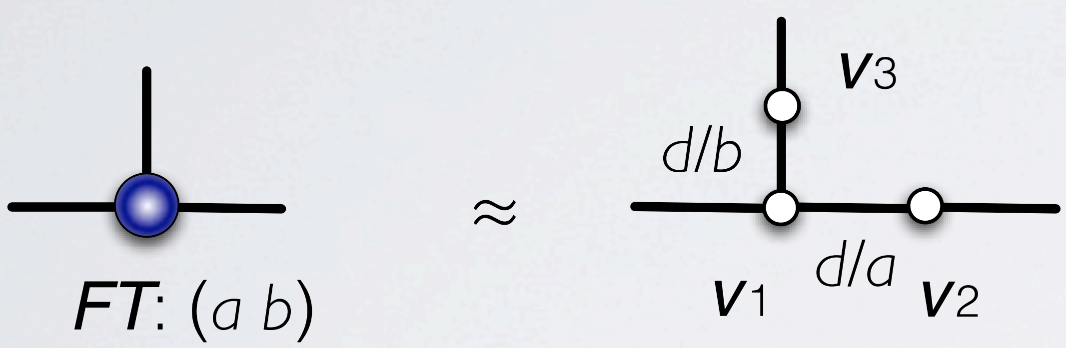
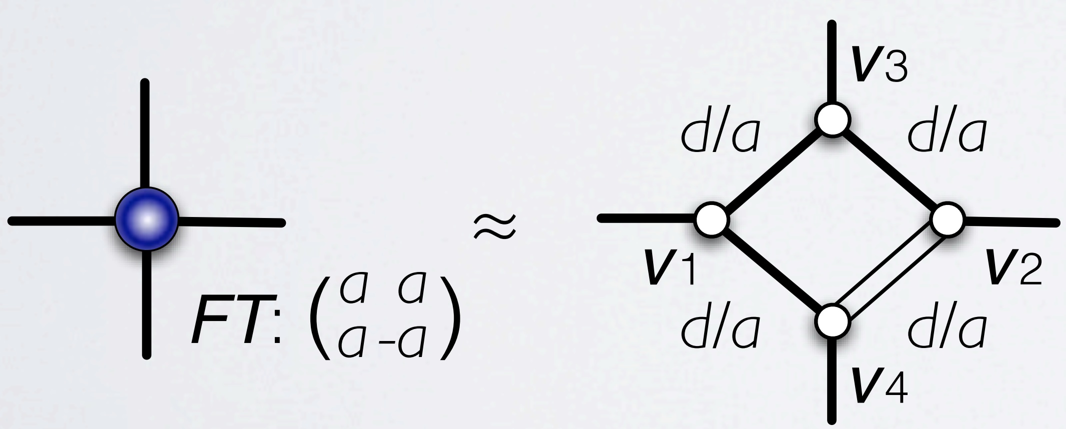
- $[0,\sqrt{U}]$ flat band-filter

FINITE GRAPH REALIZATION

- Exotic vertex from delta vertices (+magnetic field)
- 

$$v_1 = [a(a-1) + b(b-1)]/d$$
$$v_2 = (1-a)/d$$
$$v_3 = (1-b)/d$$

- 

$$v_1 = v_2 = 2a(a-1)/d$$
$$v_3 = v_4 = (1-2a)/d$$

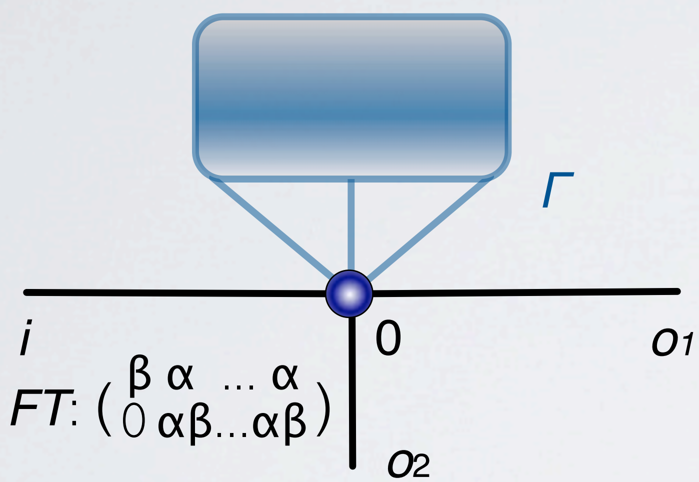
∗ No 1-»2 (interference) *U*=0

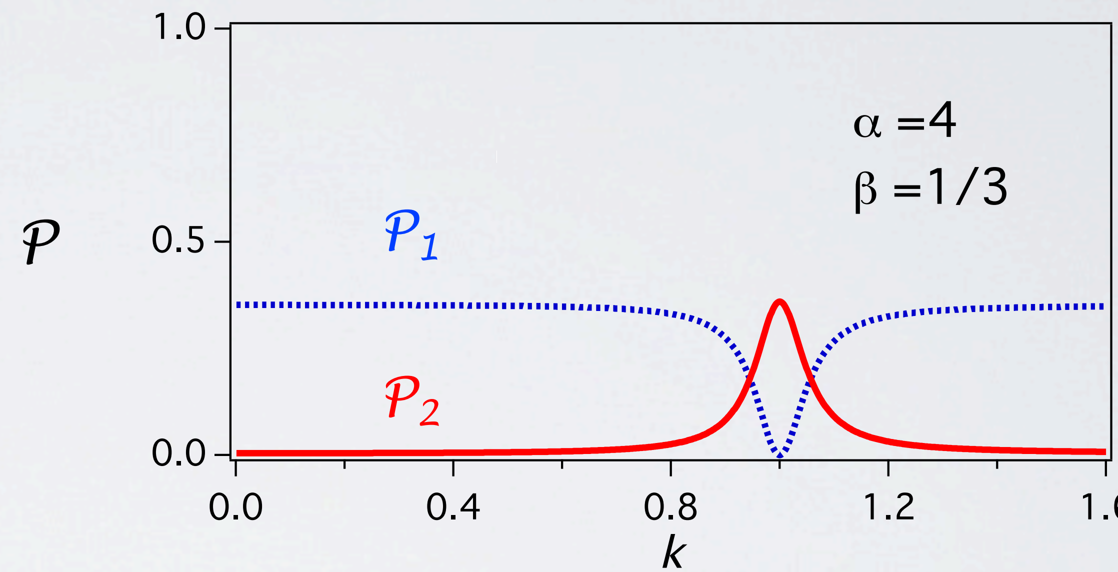
∗ No 1-»3 with added *U* > *E*

⟶ flux pushed to 1-»2

GENERALIZATIONS

- Generalized spectral filter can be designed





$$\begin{pmatrix} 1 & 0 & \beta & \alpha & \cdots & \alpha \\ 0 & 1 & 0 & \alpha\beta & \cdots & \alpha\beta \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & & & & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} -\psi'_-(0) \\ \psi'_1(0) \\ \psi'_2(0) \\ \phi'_1(0) \\ \vdots \\ \phi'_n(0) \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ -\beta & 0 & 1 & 0 & \cdots & 0 \\ -\alpha & -\alpha\beta & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \ddots & \\ -\alpha & -\alpha\beta & 0 & 0 & \cdots & 1 \end{pmatrix} \begin{pmatrix} \psi_-(0) \\ \psi_1(0) \\ \psi_2(0) \\ \phi_1(0) \\ \vdots \\ \phi_n(0) \end{pmatrix}$$

SUMMARY & PROSPECTS

- Detailed study of quantum graph vertex
  --- making sense by charting of n^2-parameter space
  --- physical realization of exotic quantum vertex

- Quantum graph with taylor-made scattering properties
  --- Hermite & unitary square matrix

- Toward designing quantum transistor

- Pedagogical value for "advanced" quantum mechanics

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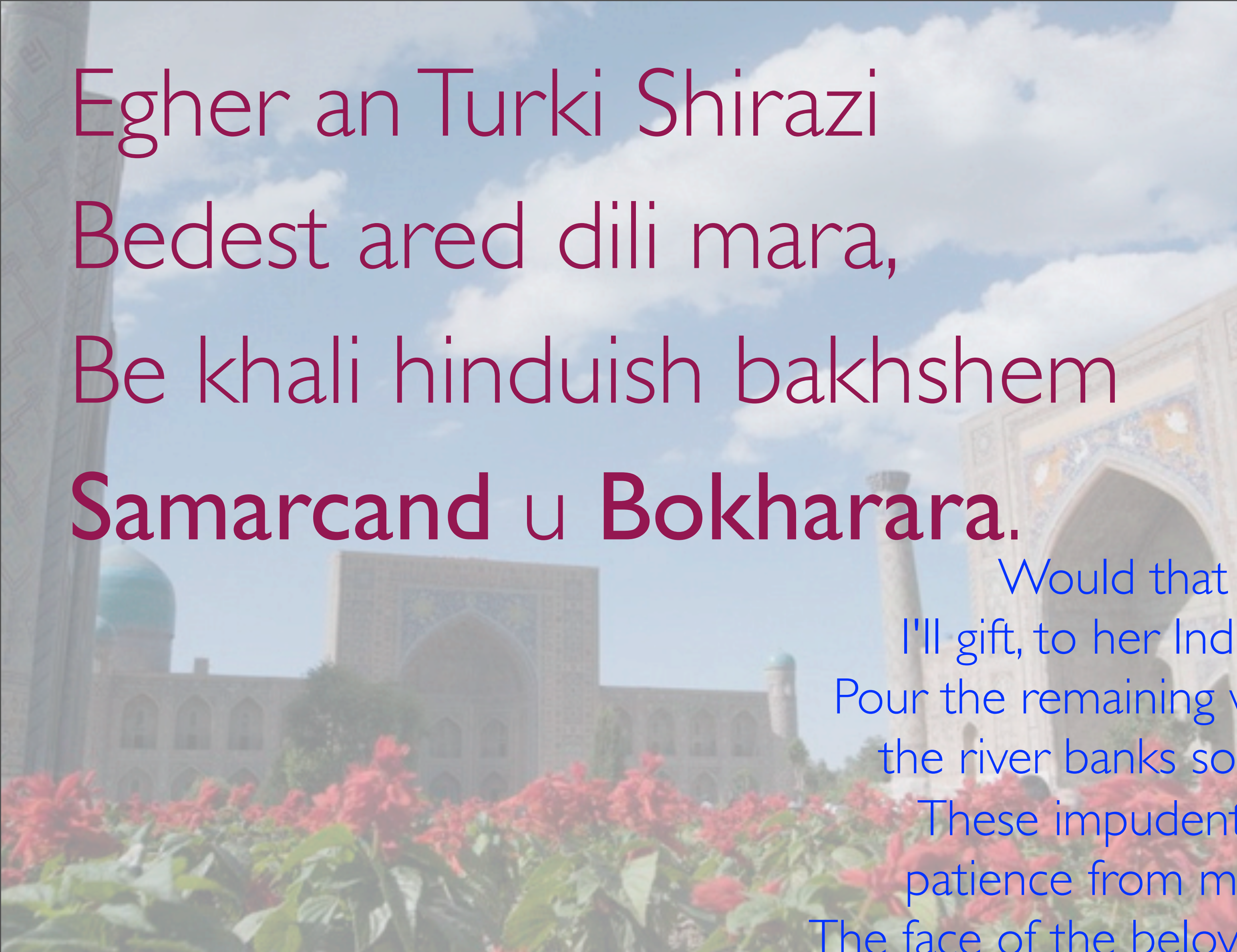

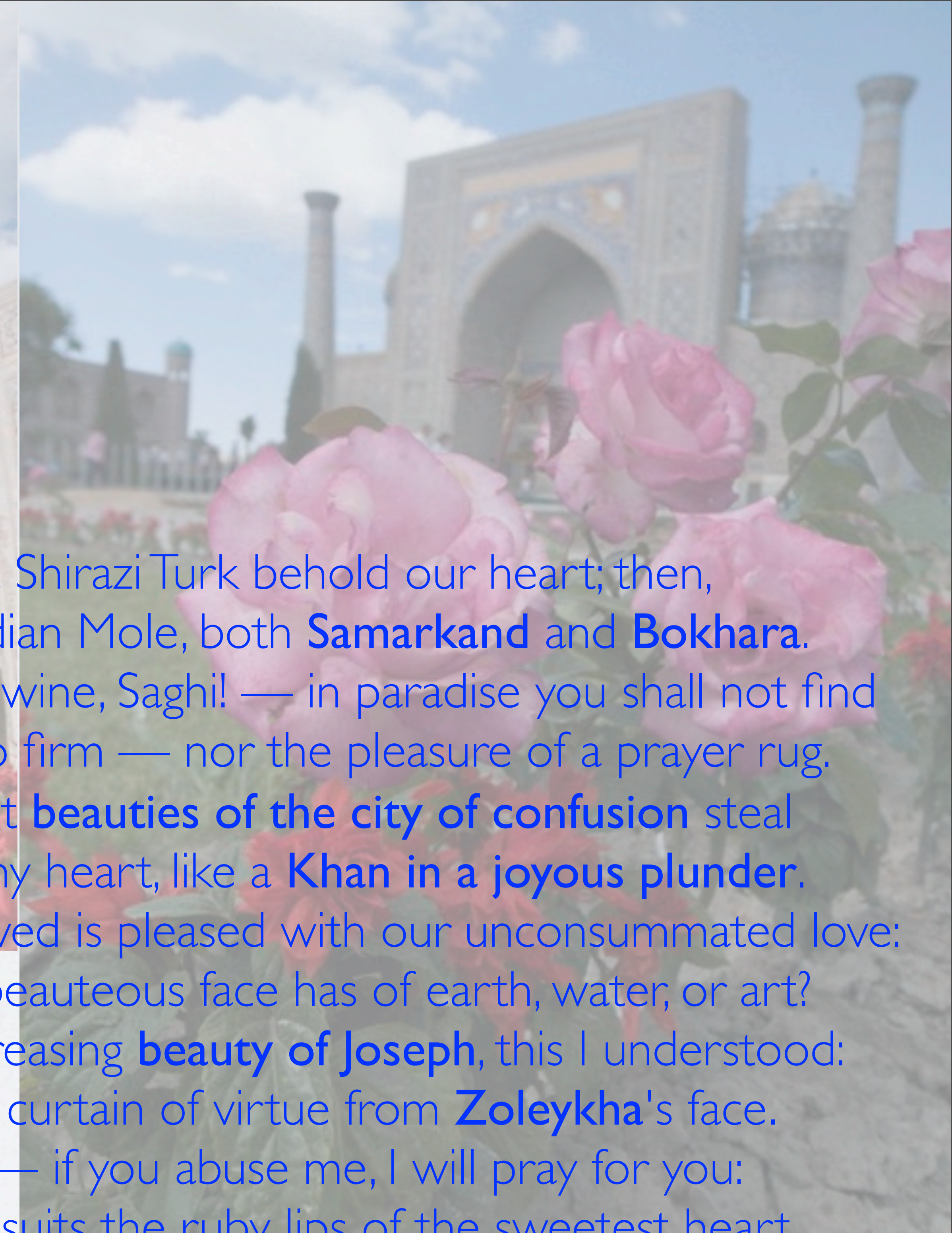

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