NLS on graphs.
(Few) Results and (many) problems

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R. A., D. Noja CMP 13
R. A., C. Cacciapuoti, D. Finco, D. Noja JPA 12, EPL 12, arXiv:1206.5201, 1211.1515

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May 20-24, 2013

Plan of the talk

1. Graphs
   - Metric graphs
   - Star graphs and related Sobolev spaces
   - NLS on star graphs

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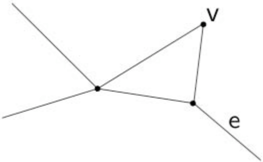
3. Absence of ground states for NLS on free graphs. A rearrangement framework.
4. Star graphs for with delta-like vertex: Orbital stability of the ground state for an attractive delta vertex.

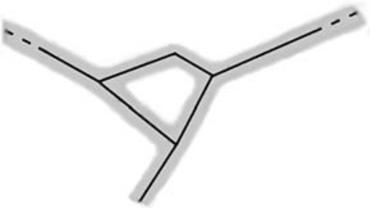
$$i\partial_t \psi(t,x) \;=\; -\Delta\psi(t,x) + V(x)\psi(t,x) + \lambda|\psi(t,x)|^{2\mu}\psi(t,x).$$

where $x \in \mathbb{R}^N$, $t \in \mathbb{R}$.

- NLS: paradigm of nonlinear wave propagation: dispersion, scattering, bound states, breathers, solitons, stability.
- Many physical systems described by NLS: Langmuir waves in plasma physics, e.m. pulse propagation in Kerr media, dynamics of BEC (Gross-Pitaevskii equation).
- NLS on graphs: Y-junctions, or more complicated structures. New subject (Burioni-Cassi-Sodano-Trombettoni-Vezzani 06-
  Bellazzini-Mintchev 08
  Gnutzmann-Smilansky-Derevyanko 11
  Sobirov-Matrasulov-Sabirov-Sawada-Nakamura 09-
  A.-Cacciapuoti-Finco-Noja 09-)

- A metric graph is a set of *edges* and *vertices* with a metric structure on any edge.
- On a metric graph, one can define differential operators.

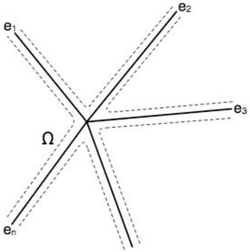




- Graphs provide one-dimensional approximations for constrained dynamics in which transverse dimensions are small with respect to longitudinal ones.

- Spectrum of valence electrons in organic molecules (Ruedenberg and Scherr 53)
- Nanotechnologies (circuits of quantum wires)
- Spectra of electromagnetic waves in thin dielectrics
- Thin acoustic waveguides

The basic element of any graph is the *star graph* $\mathcal{G}_N$



- Hilbert Space

$$\mathcal{H} = \bigoplus_{j=1}^{N} L^2((0,+\infty)) \qquad \Psi = (\psi_1, \dots, \psi_N) \in \mathcal{H}$$

- Laplacian on $\mathcal{H}$

$$-\Delta\Psi = \left(-\frac{d^2\psi_1}{ds^2}, \dots, -\frac{d^2\psi_N}{ds^2}\right)$$

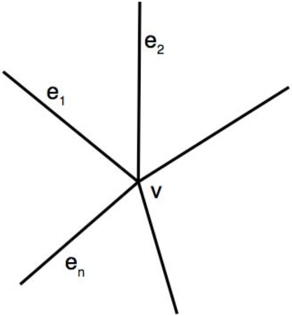
$$\mathscr{D}(-\Delta) = \bigoplus_{j=1}^{n} H^2((0,+\infty)) \quad + \quad \text{self-adjoint conditions in the vertices and endpoints of the edges}$$

Boundary conditions at the vertices

Kostrykin-Schrader 99: $-\Delta$

$$A \begin{pmatrix} \psi_1(v) \\ \vdots \\ \psi_N(v) \end{pmatrix} = B \begin{pmatrix} \psi_1'(v) \\ \vdots \\ \psi_N'(v) \end{pmatrix}$$

- A and B are $N \times N$ matrices
- $(A|B)$ has maximal rank N
- AB^* is hermitian



Decoupling (Dirichlet's) condition:

$$\psi_1(v) = \psi_2(v) = \cdots = \psi_N(v) = 0$$

Standard (Kirchhoff's) condition:

$$\psi_1(v) = \psi_2(v) = \cdots = \psi_N(v)$$

$$\sum_{j=1}^{N} \psi_j'(v) = 0$$

Non-linear focusing Schrödinger evolution

$$i\frac{d}{dt}\Psi_t = H\Psi_t - |\Psi_t|^{2\mu}\Psi_t \qquad |\Psi_t|^{2\mu}\Psi_t \equiv (|\psi_1(t)|^{2\mu}\psi_1(t), \ldots, |\psi_N(t)|^{2\mu}\psi_N(t))^T$$

where H is a Laplacian with fixed self-adjoint conditions at the vertex.

In components

$$i\frac{\partial}{\partial t}\psi_j(x_j,t) = -\frac{\partial^2}{\partial x_j^2}\psi_j(x_j,t) - |\psi_j(x_j,t)|^{2\mu}\psi_j(x_j,t) \qquad x_j > 0$$

$+$ self-adjoint boundary conditions at the vertex.

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Preliminarily:
Local (in t) well-posedness in the form domain of H for any $\mu > 0$, global well-posedness for $\mu < 2$.

Two conservation laws: L^2-norm $\|\Psi_t\|^2$ and energy

$$E(\Psi_t) := \frac{1}{2}(\Psi_t, H\Psi_t) - \frac{1}{2\mu+2}\|\Psi_t\|_{2\mu+2}^{2\mu+2}$$

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We call a ground state any function that minimizes the energy among all functions with a fixed L^2-norm.

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Ground states are stationary states, i.e., $\Psi_t^\omega(x) = e^{i\omega t}\Phi_\omega(x)$.

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A stationary state Ψ^ω is *orbitally stable* if for any $\varepsilon > 0$ there exists a $\sigma > 0$ s.t.

$$\inf_{\theta\in[0,2\pi)} \|\Psi_0 - e^{i\theta}\Psi^\omega\|_Q \leq \sigma \Rightarrow \sup_{t>0} \inf_{\theta\in[0,2\pi)} \|\Psi_t - e^{i\theta}\Psi^\omega\|_Q \leq \varepsilon$$

where Ψ_t is the solution corresponding to the initial condition Ψ_0.

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By general theory (Lions, Cazenave-Lions '80s), Ground States are orbitally stable (see also Grillakis-Shatah-Strauss '80s).

One-dimensional focusing NLS (Cazenave-Lions 82)

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Stationary solutions

$$\psi_\omega(t,x) \;=\; e^{i\theta}e^{i\omega t}(\omega(\mu+1))^{\frac{1}{2\mu}}\cosh^{-\frac{1}{\mu}}(\mu\sqrt{\omega}(x-\bar{x})),\; \omega > 0$$

with $\theta \in [0, 2\pi)$, $\bar{x} \in \mathbb{R}$.

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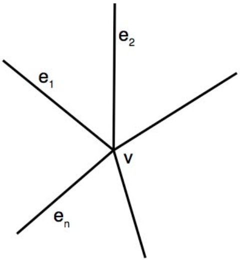
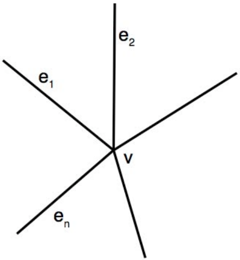
Orbitally stable for $\mu < 2$ and *orbitally unstable* for $\mu \geq 2$.

Free Graphs: Kirchhoff's conditions



Matching conditions:

$$\psi_1(0) = \psi_2(0) = \ldots = \psi_N(0)$$
$$\sum_{j=1}^{N} \psi_j'(0) = 0$$



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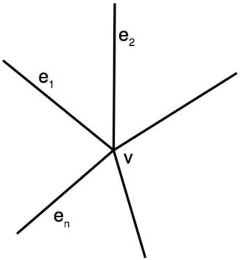
Stationary states are found by solving the stationary NLS edge by edge and by imposing the conditions at the vertex. One obtains

$$(\Psi_\omega)_i(x) = \sigma_i[(\mu+1)\omega]^{\frac{1}{2\mu}} \cosh^{-\frac{1}{\mu}}(\mu\sqrt{\omega}(x-a_i)).$$



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$$(\Psi_\omega)_i(x) = \sigma_i[(\mu+1)\omega]^{\frac{1}{2\mu}} \cosh^{-\frac{1}{\mu}}(\mu\sqrt{\omega}(x-a_i)).$$

The continuity condition then yield $\sigma_i = 1$, $a_i = \varepsilon_i a$ with $a > 0$ and $\varepsilon = \pm 1$. The condition on the derivatives translates to

$$\tanh(\mu\sqrt{\omega}a) \sum_{i=1}^{N} \varepsilon_i = 0$$

Standing waves: $\alpha = 0$ (Kirchhoff's vertex)

$$\tanh(\mu\sqrt{\omega}a)\sum_{i=1}^{N}\varepsilon_i = 0$$

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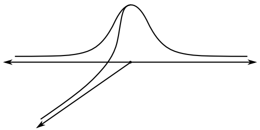
N odd: $a = 0$. The stationary state is unique and "symmetric"

$$\phi_{\omega,i}(x) = \phi_\omega(0,x) \quad i = 1,\ldots,N$$

N half solitons continuously joint at the vertex



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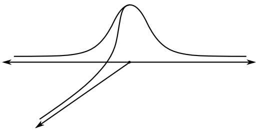
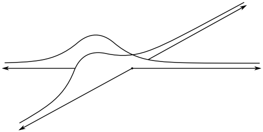


N even: $a \in \mathbb{R}$, $\sum_{i=1}^{N}\varepsilon_i = 0$ There is a one-parameter family of stationary states

$$\phi_{\omega,i}(a;x) = \begin{cases}\phi_\omega(-a,x) & i = 1,\dots N/2\\ \phi_\omega(+a,x) & i = N/2+1,\dots N\end{cases}$$

$N/2$ identical solitons. All stationary solution have the same energy



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Graphs: non existence of a minimum (ACFN 12)

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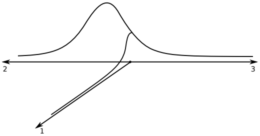
$$E(\Sigma_n) \to -\frac{M^3}{96}$$



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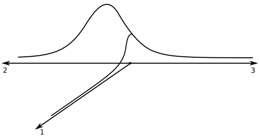
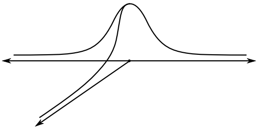
Fix $\mu = 1$ and consider the sequence of "sesquisolitons" Σ_n escaping at infinity

$$E(\Sigma_n) \to -\frac{M^3}{96} \qquad\qquad E(\Phi_\omega^0) = -\frac{M^3}{216}$$





- The unique symmetric stationary state Φ_ω^0 is not a ground state.

- Furthermore, it is not stable as a stationary state.

- What about symmetric rearrangements?

Proposition (Pólya-Szegő inequality)

Assume that $\Psi \in \mathscr{D}(E)$. Then $\|\Psi^*\|_p = \|\Psi\|_p$, for any $1 \leq p \leq \infty$, and $\|\Psi^{*\prime}\|^2 \leq \|\Psi'\|^2$

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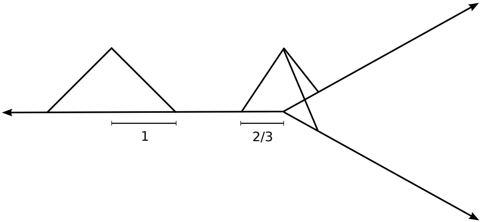


Same L^p norms, but

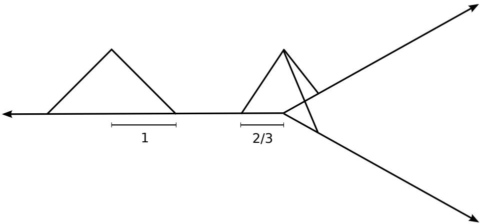
$$\|\Psi'\|^2 = 2\,; \quad \|\Psi^{*\prime}\|^2 = \frac{9}{2}$$

so that

$$\|\Psi^{*\prime}\|^2 = \frac{N^2}{4}\|\Psi'\|^2 \qquad (N = 3)$$

Proposition (Modified Pólya-Szegő inequality)

Assume that $\Psi \in \mathscr{D}(E)$. Then $\|\Psi^*\|_p = \|\Psi\|_p$, for any $1 \le p \le \infty$, and $\|\Psi^{*\prime}\|^2 \le \frac{N^2}{4}\|\Psi'\|^2$



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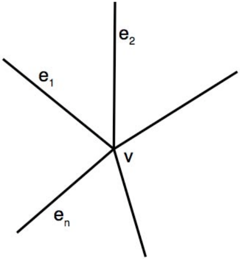

Matching conditions:

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$$\sum_{j=1}^{N} \psi_j'(0) = \alpha \psi_1(0)$$

Stationary states are found by solving the stationary NLS edge by edge and imposing the conditions at the vertex. One obtains

$$(\Psi_\omega)_i(x) = \sigma_i [(\mu+1)\omega]^{\frac{1}{2\mu}} \cosh^{-\frac{1}{\mu}}(\mu\sqrt{\omega}(x-a_i)).$$

The chosen conditions then yield $\sigma_i = 1$, $a_i = \varepsilon_i a$, where $a > 0$ and $\varepsilon_i = \pm 1$.

In particular

- $\varepsilon_i = 1$: there is a "bump" on the edge i
- $\varepsilon_i = -1$: there is a "tail" on the edge i

By the condition on the derivative

$$\tanh(\mu\sqrt{\omega}a)\sum_{i=1}^{N}\varepsilon_i = \frac{\alpha}{\sqrt{\omega}} \tag{1}$$

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$\sum_{i=1}^{N}\varepsilon_i$ must have the same sign of α

- $\alpha > 0$ strictly more bumps than tails
- $\alpha < 0$ strictly more tails than bumps
- $\alpha = 0$ same number of tails and bumps or $a = 0$

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For every configuration of $\varepsilon_1, ..., \varepsilon_N$ (up to permutations of the edges) the condition (1) fixes uniquely a

We index the stationary states with the number j of bumps.

Stationary solutions exist

$$\Phi^j_\omega = (\phi^j_{\omega,1}, ..., \phi^j_{\omega,N})^T$$

Their components are given by

$$\phi^j_{\omega,i} = \begin{cases} \phi_\omega(a^j) & i = 1, \dots, j \\ \phi_\omega(-a^j) & i = j+1, \dots, N \end{cases}$$

$$a^j = \frac{1}{\mu\sqrt{\omega}} \operatorname{arctanh}\left(\frac{\alpha}{(2j-N)\sqrt{\omega}}\right)$$

Due to the constraint linking the number of bumps and the sign of α

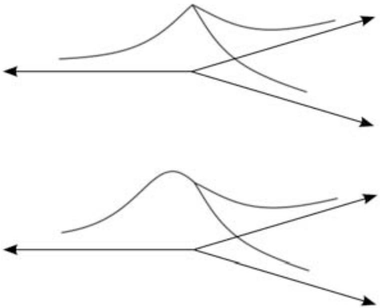
$\alpha > 0 \;:\; \Phi^j_\omega$ with $j = [N/2+1], \dots, N$

$\alpha < 0 \;:\; \Phi^j_\omega$ with $j = 0, \dots, [(N-1)/2]$

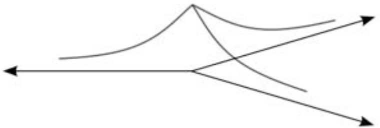
For any value of $\alpha \neq 0$ there are $\left[\frac{N+1}{2}\right]$ states

Finally, there is a lower bound on the allowed frequencies:

$$\omega > \frac{\alpha^2}{N^2}$$



Nonlinear stationary states:
$\alpha < 0\,,\ N = 3\,,\ j = 0, 1$



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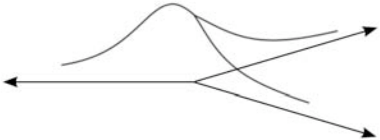

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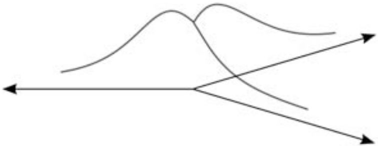

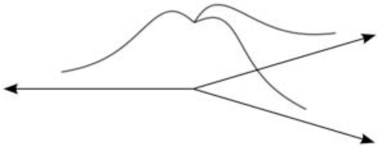

Define $\alpha^\star < 0$ by

$$\frac{2}{N}\int_0^1 (1-t^2)^{\frac{1}{\mu}}\,dt \;=\; \int_{\frac{|\alpha^\star|}{N\sqrt{\omega}}} (1-t^2)^{\frac{1}{\mu}}\,dt.$$

For $\alpha < \alpha^\star < 0$, use of symmetric decreasing rearrangements Ψ^* of Ψ to reduce the minimization problem to the set of symmetric states and prove the existence of the minimum.

Theorem

For $\mu \leq 2$, $\alpha < \alpha^*(< 0)$, the N-tail state Φ_ω^0 is an orbitally stable ground state for any $\omega > \frac{\alpha^2}{N^2}$.

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  New technique: Restricting the minimization to functions made of chunks of solitons ("multi-solitons") reduces the problem to a finite-dimensional one. Through GSS theory, it is sufficient to prove that the symmetric state is a local minimum. One can proceed as follows:
  - For any function one can construct a "multi-soliton" with the same mass but less energy.
  - The construction does not move too much from the original function.

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- Asymptotic properties of the dynamics